Condensed Matter Physics, ????, Vol. ?, No ?, ?????: 1–13

http://www.icmp.lviv.ua/journal

CONDENSED MATTER PHYSICS

Regular article

How to solve Fokker–Planck equation treating mixed eigenvalue spectrum?*

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August 24, 2012

An analogy of the Fokker–Planck equation (FPE) with the Schrödinger equation allows us to use quantum mechanics technique to find the analytical solution of the FPE in a number of cases. However, previous studies have been limited to the Schrödinger potential with discrete eigenvalue spectrum. Here we will show how this approach can be applied also for mixed eigenvalue spectrum with bounded and free states. We solve the FPE with boundaries located at $x = \pm L/2$ and take the limit $L \to \infty$, considering examples with constant Schrödinger potential and with Pöschl-Teller potential. An oversimplified approach has been earlier proposed by M.T. Araujo and E. Drigo Filho. A detailed investigation of the two examples shows that the correct solution, obtained in this paper, is consistent with the expected Fokker–Planck dynamics.

Key words: *Fokker–Planck equation, Schrödinger equation, Pöschl–Teller potential*

PACS: *05.10.Gg*

1. Introduction

The one–dimensional Fokker–Planck equation (FPE) for the probability density $p(x,t)$, depending on variable x and time t, assumes the generic form [1–7]

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{f(x,t)p(x,t)\right\} + \frac{\partial^2}{\partial x^2}\left\{\frac{D(x,t)}{2}p(x,t)\right\} . \tag{1.1}$$

Here the drift coefficient or force $f(x,t)$ and the diffusion coefficient $D(x,t)$ depend on x and t in general. The Fokker–Planck equation is related to the Smoluchowski equation. Starting from pioneering works by Marian Smoluchowski [1, 2], these equations have been historically used to describe the Brownian–like motion of particles. The Smoluchowski equation describes the high–friction limit, whereas the Fokker–Planck equation refers to the general case.

The FPE provides a very useful tool for modeling a wide variety of stochastic phenomena arising in physics, chemistry, biology, finance, traffic flow, etc. [3–6]. Given the importance of the Fokker–Planck equation, different analytical and numerical methods have been proposed for its solution. As it is well known, the stationary solution of FPE can be given in a closed form if the condition of detailed balance holds. The study of the time–dependent solution is a much more complicated problem. The FPE (1.1) with a general time–dependence and a special x–dependence of the drift and diffusion coefficients has been studied analytically in [7] using Lie algebra. This method is applicable when the Fokker–Planck equation has certain algebraic structure, allowing to apply the Lie algebra and the Wei–Norman theorem. Generally, there are few exactly solvable cases. A simple such example is a system with constant diffusion coefficient and harmonic interaction of the form $f(x) = -dV(x)/dx$ with harmonic potential $V(x) \sim x^2$. The case with double–well potential is already quite non-trivial and requires a numerical approach [8].

*The authors B. M. and J. K. thank for financial support from Academic Exchange Office at Rostock University allowing to continue our long-standing collaboration between Rostock (Germany) and Riga (Latvia).

The known relation between the Fokker–Planck equation and the Schrödinger equation can also be used. This approach allows us to apply the well known methods of quantum mechanics. In particular, analytical solutions can be found in the cases, where the eigenvalues and eigenfunctions for the considered Schrödinger potential are known. For a general Schrödinger potential, numerical treatments used in quantum mechanics, such as the Crank–Nicolson time propagation with implicit Numerov's method for second order derivatives [9], are very useful. To apply it to Schrödinger–type equation, we just need to replace the real time step Δt by an imaginary time step $\Delta t \to -i\Delta t$. In quantum mechanics this is called imaginary time propagation and is used for calculation of ground states and also excited states. The analytical studies of mapping the FPE to Schrödinger equation has been up to now restricted to a treatment of discrete eigenstates. An attempt has been made in [10] to extend this approach to potentials with mixed (discrete and continuous) eigenvalue spectrum. However, we have found a basic error in this treatment, indicated explicitly at the end Sec. 4.3.

The aim of our work is to show how the problem with mixed eigenvalue spectrum can be treated correctly. We will show this in two examples: one with constant Schrödinger potential and another – with Pöschl–Teller potential. The same example has been incorrectly treated in [10]. To avoid any confusion one has to note that the Pöschl–Teller potential is called Rosen–Morse potential in [10].

2. Solution of FPE with constant diffusion coefficient

We start our consideration with the one–dimensional Fokker–Planck equation (1.1) in the following formulation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)p(x,t)\right] + \frac{D}{2}\frac{\partial^2 p(x,t)}{\partial x^2} \tag{2.1}$$

for the probability density distribution $p(x,t)$, depending on variable x and time t. Here $f(x)$ is the nonlinear force and D is the diffusion coefficient, which is now assumed to be constant. We consider natural boundary conditions

$$\lim_{x\to\pm\infty} p(x,t) = \lim_{x\to\pm\infty} \frac{\partial p(x,t)}{\partial x} = 0 \tag{2.2}$$

and take the most frequently used initial condition

$$p(x,t=0) = \delta(x-x_0) \tag{2.3}$$

in the form of the δ–function. This FPE (2.1) can be transformed into an equation of Schrödinger type (see Sec. 2.2). Unfortunately, the well known relation (see Eq. (2.25)), derived for the discrete eigenvalue spectrum, cannot be applied if this equation has continuous or mixed eigenvalue spectrum. To overcome this problem, we follow a properly corrected treatment of [10]. Namely, we solve the FPE with boundaries located at $x = \pm L/2$ and then take the limit $L \to \infty$ (see Sec. 2.3). This approach is applied in quantum mechanics to describe unbounded states. To keep closer touch with quantum mechanics, here we will use the boundary conditions $p(x = \pm L/2, t) = 0$, further called absorbing boundaries.

2.1. The stationary solution

The stationary solution $p_{st}(x)$ is the long–time limit of $p(x,t)$ at $t \to \infty$, which follows from the equation

$$0 = \frac{d}{dx}\left[f(x)p_{st}(x)\right] - \frac{D}{2}\frac{d^2 p_{st}(x)}{dx^2}\ . \tag{2.4}$$

The force $f(x)$ can be expressed in terms of the potential $V(x)$ via $f(x) = -dV(x)/dx$. It yields

$$0 = -\frac{d}{dx}\left[\frac{dV(x)}{dx}p_{st}(x) + \frac{D}{2}\frac{dp_{st}(x)}{dx}\right]\ . \tag{2.5}$$

Due to the natural boundary conditions, we have zero flux

$$j_{st}(x) \equiv -\frac{dV(x)}{dx}p_{st}(x) - \frac{D}{2}\frac{dp_{st}(x)}{dx} = C \quad \text{with} \quad C = 0\ . \tag{2.6}$$

Thus, we have

$$\frac{dp_{st}(x)}{dx} = -\frac{2}{D}\frac{dV(x)}{dx}p_{st}(x) \tag{2.7}$$

$$\frac{dp_{st}(x)}{p_{st}(x)} = -\frac{2}{D}dV(x)\ , \tag{2.8}$$

which yields the stationary solution

$$p_{st}(x) = \mathcal{N}^{-1}Y(x)\ , \tag{2.9}$$

where

$$Y(x) \equiv \exp\left[-\frac{2}{D}V(x)\right] \tag{2.10}$$

has meaning of the unnormalized stationary solution only in case of natural boundaries and $\mathcal{N}$ is the normalization constant

$$\mathcal{N} = \int_{-\infty}^{+\infty} dx \exp\left[-\frac{2}{D}V(x)\right]\ . \tag{2.11}$$

This function $Y(x)$ is further used to construct the time–dependent solution.

2.2. The time–dependent solution with discrete eigenvalues

Here we derive the time–dependent solution, starting with the transformation $p(x,t) \to q(x,t)$ defined by

$$p(x,t) = Y(x)^{1/2}\ q(x,t) \equiv \exp\left[-\frac{2}{D}\frac{V(x)}{2}\right] q(x,t)\ . \tag{2.12}$$

This transformation removes the first derivative in the original Fokker–Planck equation and generates the equation of Schrödinger type for the function $q(x,t)$, i. e.,

$$\frac{\partial q(x,t)}{\partial t} = -V_S(x)q(x,t) + \frac{D}{2}\frac{\partial^2 q(x,t)}{\partial x^2}\ , \tag{2.13}$$

where

$$V_S(x) = -\left[\frac{1}{2}\frac{d^2V(x)}{dx^2} - \frac{2}{D}\left(\frac{1}{2}\frac{dV(x)}{dx}\right)^2\right]\ , \tag{2.14}$$

is the so–called Schrödinger potential. In the case of discrete eigenvalues, we apply the superposition ansatz

$$q(x,t) = \sum_{n=0}^{\infty} a_n(t)\psi_n(x)\,. \tag{2.15}$$

After inserting (2.15) into (2.13), we get the eigenvalue problem

$$\frac{D}{2}\frac{d^2\psi_n(x)}{dx^2} - V_S(x)\psi_n(x) = -\lambda_n\psi_n(x) \tag{2.16}$$

for eigenfunctions $\psi_n(x)$ and eigenvalues $\lambda_n \geq 0$ with time–dependent coefficients $a_n(t)$ given by

$$a_n(t) = a_n(0)\exp\left(-\lambda_n\, t\right)\ . \tag{2.17}$$

According to this, Eq. (2.15) can be written as

$$q(x,t) = \sum_{n=0}^{\infty} a_n(0) e^{-\lambda_n t} \psi_n(x) \ . \tag{2.18}$$

The eigenfunctions $\psi_n(x)$ are orthonormal, i. e.,

$$\int_{-\infty}^{+\infty} \psi_n(x)\psi_m(x) dx = \delta_{nm} \tag{2.19}$$

and satisfy the closure condition (completeness relation)

$$\sum_{n=0}^{\infty} \psi_n(x')\psi_n(x) = \delta(x - x') \ . \tag{2.20}$$

Eq. (2.16) can be written as a Schrödinger–type eigenvalue equation with Hermitian Hamilton operator $\mathcal{H}$:

$$\mathcal{H}\psi_n(x) = \lambda_n \psi_n(x) \quad \text{with} \quad \mathcal{H} = -\frac{D}{2}\frac{d^2}{dx^2} + V_S(x) \ . \tag{2.21}$$

The coefficients $a_n(0)$ in (2.18) are calculated using the initial condition

$$p(x, t = 0) = Y(x)^{1/2} q(x, t = 0) = \delta(x - x_0) \ . \tag{2.22}$$

According to (2.18), this relation can be written as

$$Y(x)^{-1/2} \delta(x - x_0) = \sum_{m=0}^{\infty} a_m(0) \psi_m(x) \ . \tag{2.23}$$

In the following, we multiply both sides of this equation by $\psi_n(x)$ and integrate over x from $-\infty$ to $+\infty$. Taking into account (2.19), it yields the up to now unknown coefficients

$$a_n(0) = Y(x_0)^{-1/2} \psi_n(x_0) \ . \tag{2.24}$$

The final result of this calculation reads

$$p(x,t) = \sqrt{\frac{Y(x)}{Y(x_0)}} \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n(x_0) \psi_n(x) \ . \tag{2.25}$$

Note that this method can also be used for other boundary conditions. The solution in the general form of (2.25) is well known from older studies, e. g., [11] and can be found in many textbooks, e. g., [3, 4].

2.3. The time–dependent solution with mixed eigenvalue spectrum

Consider now the problem with two absorbing boundaries located at $x = \pm L/2$ instead of the natural boundary conditions. In this case we have a discrete eigenvalue spectrum, and Eq. (2.25) can be used (with summation over only those eigenfunctions which satisfy boundary conditions in a box of length L) to calculate the probability distribution $p_L(x,t)$, i. e.,

$$p_L(x,t) = \sqrt{\frac{Y(x)}{Y(x_0)}} \sum_{n=0}^{\infty} e^{-\lambda_{n,L} t} \psi_{n,L}(x_0) \psi_{n,L}(x) \ , \tag{2.26}$$

where $\lambda_{n,L}$ are eigenvalues and $\psi_{n,L}(x)$ are corresponding eigenfunctions, which fulfill the boundary conditions. Let us split this infinite sum into two parts: for $\lambda_{n,L} < \lambda_{con}$ and $\lambda_{n,L} \geq \lambda_{con}$, where λ_{con}

is the smallest continuum eigenvalue in the case with natural boundaries. This eigenvalue spectrum is shown schematically in Fig. 1, where the value of λ_{con} is indicated by a horizontal dotted line, the eigenvalues $\lambda_{n,L} < \lambda_{con}$ – by solid lines and the eigenvalues $\lambda_{n,L} \geq \lambda_{con}$ – by dashed lines. Let $M(L)$ be maximal value of n for which $\lambda_{n,L} < \lambda_{con}$ and $k_{n-M(L),L} = \sqrt{\frac{2}{D}(\lambda_{n,L} - \lambda_{con})}$ for $n > M(L)$ and $\psi^{con}_{k_{n-M(L),L}}(x) = \psi_{n,L}(x)$ for $n > M(L)$. Hence, we have

$$
\begin{aligned}
p_L(x,t) = {} & \sqrt{\frac{Y(x)}{Y(x_0)}} \sum_{n=0}^{M(L)} e^{-\lambda_{n,L}\, t} \psi_{n,L}(x_0) \psi_{n,L}(x) \\
& + \sqrt{\frac{Y(x)}{Y(x_0)}} e^{-\lambda_{con}\, t} \sum_{m=1}^{\infty} e^{-\frac{D}{2} k^2_{m,L}\, t} \psi^{con}_{k_{m,L}}(x_0) \psi^{con}_{k_{m,L}}(x) \; .
\end{aligned}
\tag{2.27}
$$

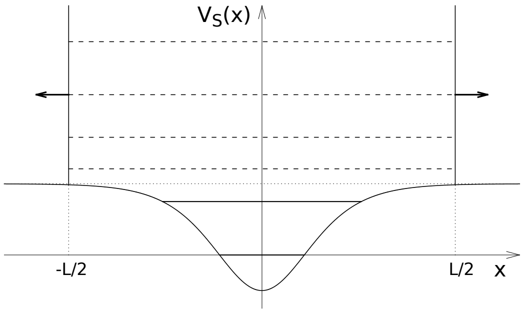

Figure 1. A schematic view of the eigenvalue spectrum for the problem with two absorbing boundaries at $x = \pm L/2$. The Schrödinger potential $V_S(x)$ together with the boundaries at $x = \pm L/2$ is indicated by solid curve and vertical lines.

The solution with natural boundaries is the limit case $L \to \infty$

$$
p(x,t) = \lim_{L\to\infty} p_L(x,t)
\tag{2.28}
$$

or

$$
\begin{aligned}
p(x,t) = {} & \sqrt{\frac{Y(x)}{Y(x_0)}} \sum_{n=0}^{N-1} e^{-\lambda_n\, t} \psi_n(x_0) \psi_n(x) \\
& + \sqrt{\frac{Y(x)}{Y(x_0)}} e^{-\lambda_{con}\, t} \lim_{L\to\infty} \sum_{m=1}^{\infty} e^{-\frac{D}{2} k^2_{m,L}\, t} \psi^{con}_{k_{m,L}}(x_0) \psi^{con}_{k_{m,L}}(x) \; ,
\end{aligned}
\tag{2.29}
$$

where $N = \lim\limits_{L\to\infty} M(L)$ is number of bounded states in the case with natural boundaries. Since the eigenfunctions cannot be normalized at $L \to \infty$, it is appropriate to write Eq. (2.29) for unnormalized eigenfunctions $\bar{\psi}^{con}_{k_{m,L}}(x)$,

$$
\begin{aligned}
p(x,t) = {} & \sqrt{\frac{Y(x)}{Y(x_0)}} \sum_{n=0}^{N-1} e^{-\lambda_n\, t} \psi_n(x_0) \psi_n(x) \\
& + \sqrt{\frac{Y(x)}{Y(x_0)}} e^{-\lambda_{con}\, t} \lim_{L\to\infty} \sum_{m=1}^{\infty} e^{-\frac{D}{2} k^2_{m,L}\, t} \underbrace{\frac{\mathcal{N}^{-1}}{\Delta k_L}}_{g^{-1}(k,L)} \bar{\psi}^{con}_{k_{m,L}}(x_0) \bar{\psi}^{con}_{k_{m,L}}(x) \Delta k_L \; ,
\end{aligned}
\tag{2.30}
$$

where the normalization constant $\mathcal{N}$ is given by

$$
\mathcal{N} = \int\limits_{-L/2}^{L/2} \mathrm{d}x \, |\bar{\psi}^{con}_k(x)|^2
\tag{2.31}
$$

and the expression under infinite sum is divided and multiplied by $\Delta k_L = k_{m+1,L} - k_{m,L}$.

The infinite sum can be split in two parts: one with odd m and the other – with even m. If the Schrödinger potential is symmetric, then one of these two parts contains only odd eigenfunctions $\bar{\psi}^o_k(x)$, whereas the other part – only even eigenfunctions $\bar{\psi}^e_k(x)$. In the limit $L \to \infty$, these two sums can be represented by corresponding integrals, yielding

$$
\begin{aligned}
p(x,t) &= \sqrt{\frac{Y(x)}{Y(x_0)}} \sum_{n=0}^{N-1} e^{-\lambda_n t} \psi_n(x_0)\psi_n(x) \\
&+ \sqrt{\frac{Y(x)}{Y(x_0)}} e^{-\lambda_{con} t} \int_0^\infty \mathrm{d}k \, e^{-\frac{D}{2}k^2 t} g_e^{-1}(k) \bar{\psi}^e_k(x_0)\bar{\psi}^e_k(x) \\
&+ \sqrt{\frac{Y(x)}{Y(x_0)}} e^{-\lambda_{con} t} \int_0^\infty \mathrm{d}k \, e^{-\frac{D}{2}k^2 t} g_o^{-1}(k) \bar{\psi}^o_k(x_0)\bar{\psi}^o_k(x) \,,
\end{aligned}
\tag{2.32}
$$

where

$$
g_e(k) = \lim_{L\to\infty} \left[2\Delta k_L \int_{-L/2}^{L/2} \mathrm{d}x \, |\bar{\psi}^e_k(x)|^2 \right] \tag{2.33}
$$

$$
g_o(k) = \lim_{L\to\infty} \left[2\Delta k_L \int_{-L/2}^{L/2} \mathrm{d}x \, |\bar{\psi}^o_k(x)|^2 \right] . \tag{2.34}
$$

This representation is useful if the eigenvalues and eigenfunctions are known.

3. The analytical solution of FPE with constant force

Let us consider a constant force term. In this case the Fokker–Planck equation (2.1) reads

$$
\frac{\partial p(x,t)}{\partial t} = -v_{\mathrm{drift}} \frac{\partial p(x,t)}{\partial x} + \frac{D}{2} \frac{\partial^2 p(x,t)}{\partial x^2} \, . \tag{3.1}
$$

This is a drift–diffusion problem for the potential $V(x) = -v_{\mathrm{drift}} x$ normalized to $V(x=0)=0$. No stationary solution exists for this problem, because the normalization constant $\mathcal{N}$ in Eq. (2.11) diverges in this case. Nevertheless, the transformation (2.12) $p(x,t) = Y(x)^{1/2} q(x,t)$ with

$$
Y(x) = \exp\left[-\frac{2}{D} \frac{V(x)}{2} \right] = \exp\left[\frac{v_{\mathrm{drift}}}{D} x \right] \tag{3.2}
$$

can be used here to obtain an equation of Schrödinger type (2.13) with constant Schrödinger potential

$$
V_S = \frac{1}{2D} v_{\mathrm{drift}}^2 \, . \tag{3.3}
$$

The corresponding to (2.21) stationary Schrödinger–type equation reads

$$
\frac{d^2 \psi_n(x)}{dx^2} - \left[\frac{v_{\mathrm{drift}}^2}{D^2} - \frac{2}{D}\lambda_n \right] \psi_n(x) = 0 \, . \tag{3.4}
$$

Let us now add two absorbing boundaries located at $x = \pm L/2$, where $\psi(x = \pm L/2) = 0$.

Only in the case of real $k_n = \sqrt{\frac{2}{D}\lambda_n - \frac{v_{\mathrm{drift}}^2}{D^2}} > 0$ Eq. (3.4) has non-trivial solutions

$$
\psi_n(x) = A \cos(k_n x) + B \sin(k_n x) \, , \tag{3.5}
$$

which satisfy boundary conditions. These solutions are

$$\psi_{n,L}(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos\left(k_{n,L}x\right) & \text{if } n \text{ is even} \\ \sqrt{\frac{2}{L}}\sin\left(k_{n,L}x\right) & \text{if } n \text{ is odd} \end{cases}, \tag{3.6}$$

where $n = 0, 1, 2, \ldots$ and

$$k_{n,L} = \frac{\pi}{L}(n+1). \tag{3.7}$$

According to (3.6)–(3.7), we have from (2.33) and (2.34)

$$g_e(k) = g_o(k) = \pi\,. \tag{3.8}$$

Taking into account that

$$\lambda_{con} = \lim_{L\to\infty}\min\{\lambda_{n,L}\} = \lim_{L\to\infty}\min\left\{\frac{D}{2}k_{n,L}^2 + \frac{v_{\text{drift}}^2}{2D}\right\} = \frac{v_{\text{drift}}^2}{2D} \tag{3.9}$$

holds, we obtain from Eq. (2.32) the expression

$$\begin{aligned} p(x,t) = {} & \exp\left[\frac{1}{D}v_{\text{drift}}(x-x_0)\right]\exp\left[-\frac{v_{\text{drift}}^2}{2D}\,t\right] \\ & \times\frac{1}{\pi}\int_0^\infty \mathrm{d}k\; e^{-\frac{Dk^2t}{2}}\left[\cos(kx)\cos(kx_0)+\sin(kx)\sin(kx_0)\right]. \end{aligned} \tag{3.10}$$

Using the well known identities

$$\cos(kx)\cos(kx_0)+\sin(kx)\sin(kx_0) = \cos[k(x-x_0)] \tag{3.11}$$

and

$$\int_0^\infty \mathrm{d}k\; e^{-\alpha k^2}\cos(\beta k) = \sqrt{\frac{\pi}{4\alpha}}e^{-\frac{\beta^2}{4\alpha}}\,, \tag{3.12}$$

we obtain after a simplification the well known result

$$p(x,t) = \frac{1}{\sqrt{2Dt}}\exp\left[-\frac{(x-x_0-v_{\text{drift}}t)^2}{2Dt}\right]\,, \tag{3.13}$$

which describes a moving and broadening Gaussian profile.

4. Fokker–Planck dynamics with Pöschl–Teller potential

Here we consider as a particular example the force

$$f(x) = -b\tanh\left(\alpha x\right) \tag{4.1}$$

with some positive constants b and α. This corresponds to the diffusion problem in the potential

$$V(x) = \frac{b}{\alpha}\ln\left(\cosh\alpha x\right)\,, \tag{4.2}$$

normalized to $V(x=0)=0$.

Fig. 2 shows that this potential is actually a smoothed version of the V-shaped potential. The corresponding Schrödinger potential in this case is

$$V_S(x) = \frac{b^2}{2D} - \left(\frac{b^2}{2D}+\frac{b\alpha}{2}\right)\frac{1}{\cosh^2(\alpha x)}\,. \tag{4.3}$$

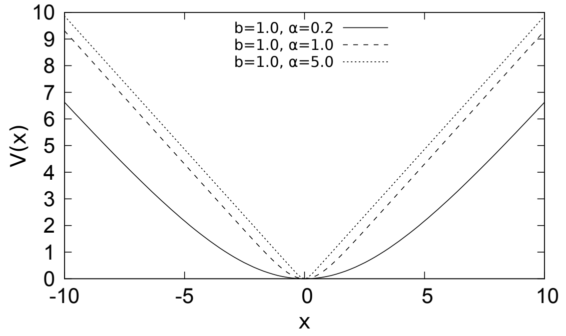

Figure 2. Graphical representation of Eq. (4.2) for $b = 1$ and several values of parameter α.

If we compare it (see Eq. (4.4) and Fig. 3) with the well known Pöschl–Teller potential

$$V_{PT}(x) = V_S(x) - \frac{b^2}{2D} = -\frac{V_0}{\cosh^2(\alpha x)} \,, \tag{4.4}$$

then we see that Eq. (4.3) represents the shifted by $\frac{b^2}{2D}$ Pöschl–Teller potential with $V_0 = \frac{b^2}{2D} + \frac{b\alpha}{2}$. As we can see from Fig. 3, the Pöschl–Teller potential gives mixed (discrete and continuous) eigenvalue spectrum, therefore Eq. (2.25) cannot be directly applied to solve the FPE. We have to use (2.32).

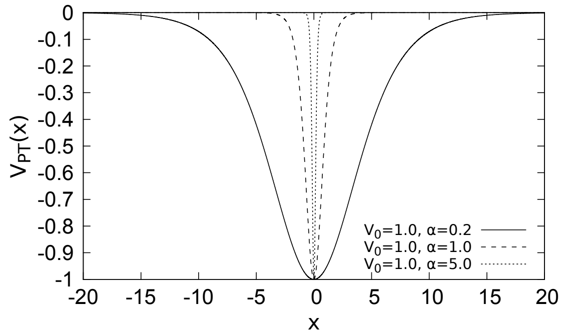

Figure 3. Pöschl–Teller potential (4.4) for $V_0 = 1$ and several values of parameter α.

The eigenvalue equation (2.16) for the potential (4.3) reads

$$\frac{D}{2}\frac{d^2\psi_n(x)}{dx^2} - \left[\frac{b^2}{2D} - \left(\frac{b^2}{2D} + \frac{b\alpha}{2}\right)\frac{1}{\cosh^2(\alpha x)}\right]\psi_n(x) = -\lambda_n\psi_n(x) \,. \tag{4.5}$$

By introducing dimensionless variables $\tilde{x} = \alpha x$, $\tilde{l} = \frac{b}{D\alpha}$ and $\tilde{\lambda}_n = \frac{2\lambda_n}{D\alpha^2} - \tilde{l}^2$, we write (4.5) in a dimensionless form

$$-\frac{d^2\psi_n(\tilde{x})}{d\tilde{x}^2} - \tilde{l}\left(\tilde{l}+1\right)\frac{1}{\cosh^2\tilde{x}}\psi_n(\tilde{x}) = \tilde{\lambda}_n\psi_n(\tilde{x}) \,. \tag{4.6}$$

Analytical solutions for bounded as well as unbounded eigenfunctions of Eq. (4.6) are known and can be found in [12, 13].

4.1. Bounded solutions for Pöschl–Teller potential

The Eq. (4.6) has $N = \max\{m \in \mathbb{N} \mid m < \tilde{l}+1\}$ bounded states $n = 0, 1, 2, \ldots, N-1$, where $\mathbb{N}$ is a natural number $\mathbb{N} = \{0, 1, 2, \ldots\}$. Here we consider the eigenfunctions with $\tilde{\lambda}_n = 0$ as unbounded, because they cannot be normalized.

The eigenvalues can be calculated from the following equation [12]

$$\tilde{\lambda}_n = -(\tilde{l} - n)^2\,, \quad \text{for } n < N\,; \quad n \in \mathbb{N}\,. \tag{4.7}$$

Note that at least one bounded state with $\tilde{\lambda}_0 = -\tilde{l}^2$ always exists for $\tilde{l} > 0$, which corresponds to $\lambda_0 = 0$. The bounded eigenfunctions are known [12]

$$\psi_n\,(\tilde{x}) = \cosh^{-\tilde{l}}(\tilde{x}) \times \begin{cases} \mathcal{N}_e(n)\,\mathrm{F}\big(-\frac{1}{2}n, \frac{1}{2}n - \tilde{l}; \frac{1}{2}; -\sinh^2\tilde{x}\big) & \text{if } n \text{ is even} \\ \mathcal{N}_o(n)\sinh(\tilde{x})\,\mathrm{F}\big(\frac{1}{2} - \frac{n}{2}, \frac{n}{2} + \frac{1}{2} - \tilde{l}; \frac{3}{2}; -\sinh^2\tilde{x}\big) & \text{if } n \text{ is odd,} \end{cases} \tag{4.8}$$

where F denotes hypergeometric function, which can be represented by Gaussian hypergeometric series

$$\mathrm{F}(\alpha, \beta; \gamma; \zeta) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + k)\Gamma(\beta + k)}{\Gamma(\gamma + k)} \frac{\zeta^n}{n!}\,. \tag{4.9}$$

The normalization constants are

$$\mathcal{N}_e(n) = \sqrt{\frac{2(\tilde{l} - n)}{(\tilde{l} - \frac{1}{2}n)\,(n+1)} \frac{1}{\mathrm{B}\big(\frac{1}{2}, \tilde{l} - \frac{1}{2}n\big)\mathrm{B}\big(\frac{1}{2}, 1 + \frac{1}{2}n\big)}}\,, \tag{4.10}$$

$$\mathcal{N}_o(n) = \sqrt{\frac{2(\tilde{l} - n)}{\tilde{l} - \frac{1}{2}\,(n+1)} \frac{1}{\mathrm{B}\big(\frac{3}{2}, \tilde{l} - \frac{1}{2}(n+1)\big)\mathrm{B}\big(\frac{1}{2}, \frac{1}{2}(n+1)\big)}}\,, \tag{4.11}$$

where $\mathrm{B}(a, b)$ is the beta function $\mathrm{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$.

4.2. Unbounded solutions for Pöschl–Teller potential

The unbounded solutions have continuous eigenvalue spectrum with $0 \le \tilde{\lambda} < \infty$. Thus we can introduce $\tilde{k} = \sqrt{\tilde{\lambda}}$ (with $\tilde{k} = k/\alpha$). The Pöschl-Teller potential is symmetric, therefore the eigenfunctions are even and odd functions known from [13]

$$\bar{\psi}_{\tilde{k},\tilde{l}}\,(\tilde{x}) = A \cdot \psi^e_{\tilde{k},\tilde{l}}\,(\tilde{x}) + B \cdot \psi^o_{\tilde{k},\tilde{l}}\,(\tilde{x})\,, \tag{4.12}$$

$$\bar{\psi}^e_{\tilde{k},\tilde{l}}\,(\tilde{x}) = (\cosh\tilde{x})^{\tilde{l}+1}\,\mathrm{F}\Big(r, s; \frac{1}{2}; -\sinh^2\tilde{x}\Big)\,, \tag{4.13}$$

$$\psi^o_{\tilde{k},\tilde{l}}\,(\tilde{x}) = (\cosh\tilde{x})^{\tilde{l}+1}\sinh(\tilde{x})\,\mathrm{F}\Big(r + \frac{1}{2}, s + \frac{1}{2}; \frac{3}{2}; -\sinh^2\tilde{x}\Big)\,, \tag{4.14}$$

where A and B are constants, and

$$r = \frac{1}{2}(\tilde{l} + 1 + i\tilde{k})\,, \quad s = \frac{1}{2}(\tilde{l} + 1 - i\tilde{k})\,. \tag{4.15}$$

Because these are unbounded solutions, eigenfunctions cannot be normalized within $x \in (-\infty; +\infty)$.

As we see, the eigenfunctions are rather complicated in general case. The expressions become essentially simpler for integer values of $\tilde{l}$. Therefore, without loosing the general idea, we will show the solutions of the Fokker-Planck equation for $\tilde{l} = 1$ and $\tilde{l} = 2$.

4.3. The solution of FPE for Pöschl–Teller potential with parameter $\tilde{l} = 1$

For $\tilde{l} = 1$ (which implies $b = \alpha D$) we have only one bounded state with the eigenvalue $\tilde{\lambda}_0 = -1$ and the eigenfunction (Eq. (4.8) for $n = 0$)

$$\psi_0(\tilde{x}) = \frac{1}{\sqrt{2}\cosh(\tilde{x})}\,. \tag{4.16}$$

The unbounded eigenfunctions (4.13) and (4.14) are

$$\bar{\psi}^e_{\tilde{k}}(\tilde{x}) = \cos(\tilde{k}\tilde{x}) - \frac{1}{\tilde{k}}\tanh(\tilde{x})\sin(\tilde{k}\tilde{x})\,, \tag{4.17}$$

$$\bar{\psi}^o_{\tilde{k}}(\tilde{x}) = \sin(\tilde{k}\tilde{x}) + \frac{1}{\tilde{k}}\tanh(\tilde{x})\cos(\tilde{k}\tilde{x})\,. \tag{4.18}$$

As proposed in Sec. 2, we add two absorbing boundaries located at $\tilde{x} = \pm\tilde{L}/2$. Due to these boundary conditions, we have only discrete values of $\tilde{k}$. Let us denote them by $\tilde{k}_{\tilde{L},m}$ for even functions and by $\tilde{\kappa}_{\tilde{L},m}$ – for odd functions. The values of $\tilde{k}_{\tilde{L},m}$ and $\tilde{\kappa}_{\tilde{L},m}$, obtained from the boundary conditions, are positive solutions of the transcendent equations

$$\tilde{k}_{\tilde{L},m} = \tanh(\tilde{L}/2)\tan(\tilde{k}_{\tilde{L},m}\tilde{L}/2), \tag{4.19}$$

$$\tilde{\kappa}_{\tilde{L},m}\tan(\tilde{\kappa}_{\tilde{L},m}) = -\tanh(\tilde{L}/2), \tag{4.20}$$

where $m = 1, 2, 3, \ldots$ denotes m-th smallest positive solution. The equations for normalized eigenfunctions now read as

$$\psi^e_{\tilde{k}_{\tilde{L},m}}(\tilde{x}) = \mathcal{N}_e^{-1/2}(\tilde{k}_{\tilde{L},m}, \tilde{L}) \cdot \left[\cos(\tilde{k}_{\tilde{L},m}\tilde{x}) - \frac{1}{\tilde{k}_{\tilde{L},m}}\tanh(\tilde{x})\sin(\tilde{k}_{\tilde{L},m}\tilde{x})\right], \tag{4.21}$$

$$\psi^0_{\tilde{\kappa}_{\tilde{L},m}}(\tilde{x}) = \mathcal{N}_o^{-1/2}(\tilde{\kappa}_{\tilde{L},m}, \tilde{L}) \cdot \left[\sin(\tilde{\kappa}_{\tilde{L},m}\tilde{x}) + \frac{1}{\tilde{\kappa}_{\tilde{L},m}}\tanh(\tilde{x})\cos(\tilde{\kappa}_{\tilde{L},m}\tilde{x})\right], \tag{4.22}$$

where normalization constants for odd and even eigenfunctions are

$$\mathcal{N}_e(\tilde{k}, \tilde{L}) = \frac{\left(\tilde{k}^2 + 1\right)\left(\tilde{k}\tilde{L} - \sin(\tilde{k}\tilde{L})\right)}{2\tilde{k}^3}\,, \tag{4.23}$$

$$\mathcal{N}_o(\tilde{k}, \tilde{L}) = \frac{\left(\tilde{k}^2 + 1\right)\left(\tilde{k}\tilde{L} + \sin(\tilde{k}\tilde{L})\right)}{2\tilde{k}^3}\,. \tag{4.24}$$

In the limit case $\tilde{L} \to \infty$, Eqs. (4.19)–(4.20) for the allowed $\tilde{k}$ values, as well as Eqs. (4.23)–(4.24) for the normalization constants simplify to

$$\tilde{k}_{\tilde{L}\to\infty,m} = \frac{2m\pi}{\tilde{L}}\,, \quad \Delta\tilde{k}_{\tilde{L}\to\infty} = \frac{2\pi}{L}\,, \tag{4.25}$$

$$\tilde{\kappa}_{\tilde{L}\to\infty,m} = \frac{(2m-1)\pi}{\tilde{L}}\,, \quad \Delta\tilde{\kappa}_{\tilde{L}\to\infty} = \frac{2\pi}{L}\,, \tag{4.26}$$

$$\mathcal{N}_e(\tilde{k}, \tilde{L} \to \infty) = \mathcal{N}_o(\tilde{k}, \tilde{L} \to \infty) = \frac{L}{2}\frac{\tilde{k}^2 + 1}{\tilde{k}^2}\,, \tag{4.27}$$

and we have also

$$g_e(\tilde{k}) = \Delta\tilde{k}_{\tilde{L}\to\infty} \cdot \mathcal{N}_e(\tilde{k}, \tilde{L} \to \infty) = \pi\frac{\tilde{k}^2 + 1}{\tilde{k}^2}\,, \tag{4.28}$$

$$g_o(\tilde{\kappa}) = \Delta\tilde{\kappa}_{\tilde{L}\to\infty} \cdot \mathcal{N}_o(\tilde{\kappa}, \tilde{L} \to \infty) = \pi\frac{\tilde{\kappa}^2 + 1}{\tilde{\kappa}^2}\,. \tag{4.29}$$

Inserting these relations and also $\lambda_{con} = \tilde{l}^2\alpha^2 D/2$ (following from $\tilde{\lambda}_{con} = \frac{2\lambda_{con}}{D\alpha^2} - \tilde{l}^2 = 0$) into (2.32), we finally obtain the time–dependent solution of the Fokker–Planck equation

$$\begin{aligned} p(x,t) =& \frac{1}{2\cosh^2(\alpha x)} \\ &+ \frac{\cosh(\alpha x_0)}{\pi\cosh(\alpha x)} e^{-\frac{D\alpha^2 t}{2}} \int\limits_0^\infty \mathrm{d}\tilde{k}\, e^{-\frac{D\alpha^2\tilde{k}^2 t}{2}} \frac{\tilde{k}^2}{\tilde{k}^2+1} \bar{\psi}^e_{\tilde{k}}(\alpha x)\, \bar{\psi}^e_{\tilde{k}}(\alpha x_0) \\ &+ \frac{\cosh(\alpha x_0)}{\pi\cosh(\alpha x)} e^{-\frac{D\alpha^2 t}{2}} \int\limits_0^\infty \mathrm{d}\tilde{k}\, e^{-\frac{D\alpha^2\tilde{k}^2 t}{2}} \frac{\tilde{k}^2}{\tilde{k}^2+1} \bar{\psi}^o_{\tilde{k}}(\alpha x)\, \bar{\psi}^o_{\tilde{k}}(\alpha x_0)\ . \end{aligned} \tag{4.30}$$

If the initial condition is given by $x_0 = 0$, then $\psi^o_{\tilde{k}}(0) = 0$ and $\psi^e_{\tilde{k}}(0) = 1$ hold, which allows us to obtain a simpler expression

$$\begin{aligned} p(x,t) =& \frac{1}{2\cosh^2(\alpha x)} + \frac{1}{\pi\cosh(\alpha x)} e^{-\frac{D\alpha^2 t}{2}} \\ &\times \int\limits_0^\infty \mathrm{d}\tilde{k}\, e^{-\frac{D\alpha^2\tilde{k}^2 t}{2}} \frac{\tilde{k}^2}{\tilde{k}^2+1} \left[\cos(\tilde{k}\alpha x) - \frac{1}{\tilde{k}}\tanh(\alpha x)\sin(\tilde{k}\alpha x)\right]\ . \end{aligned} \tag{4.31}$$

The solution for parameters $b = 2$, $D = 2$ and $\alpha = 1$, corresponding to $\tilde{l} = 1$, with the initial location of the delta–peak at $x_0 = 5$ is shown in Fig. 4 for different time moments t. As we

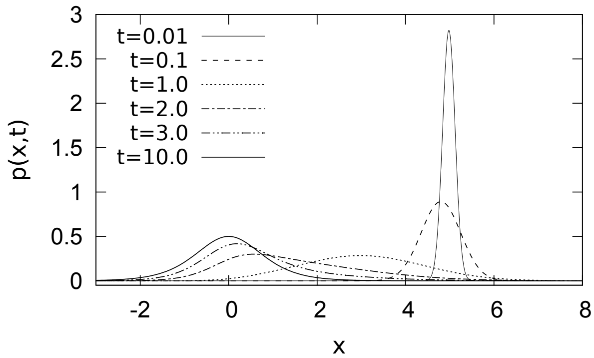

Figure 4. The probability distribution at different time moments t, calculated for the parameters $b = 2$, $D = 2$ and $\alpha = 1$ ($\tilde{l} = 1$) starting at $x_0 = 5$.

can see, the probability distribution moves to the left. It broadens at the beginning. For larger times, it becomes narrower again and converges to the stationary solution $p_{st}(x) = \lim\limits_{t\to\infty} p(x,t) = \frac{1}{\cosh^2(\alpha x)} = \psi_0(x)^2$ (see Eqs. (4.30) and (4.16)), which is a symmetric distribution around $x = 0$. The stationary solution is practically reached at $t = 10$. This behavior is expected from the drift–diffusion dynamics.

For small times $t \to 0$, we have a delta-peak located at $x = x_0$ in accordance with the given initial condition (2.3). For comparison, the "general solution" of [10] does not satisfy this initial condition, as a result of a wrong construction, where the contribution of bounded states is simply summed up with a Gaussian probability density profile (calculated with an error). The latter corresponds to unbounded states for zero Schrödinger potential at $L \to \infty$, as it is evident from (3.13) and (3.3) at $v_{\mathrm{drift}} = 0$. Therefore also the result appears to be correct only at $t \to \infty$ when the Gaussian part vanishes. It is clear that the whole set of eigenfunctions must be calculated self-consistently for the given potential to obtain a correct and meaningful result, since only in this case the completeness relation (2.20) holds and all different eigenfunctions are orthogonal.

Thus, the basic error of [10] is that some of the eigenfunctions are calculated for zero Schrödinger potential in [10], whereas all of them must be calculated for the true Schrödinger potential.

4.4. The solution of FPE for Pöschl–Teller potential with parameter $\tilde{l} = 2$

For $\tilde{l} = 2$ (which implies $b = 2\alpha D$) we have two bounded states with eigenvalues $\tilde{\lambda}_0 = -4$ and $\tilde{\lambda}_1 = -1$. The corresponding eigenfunctions are

$$\psi_0(\tilde{x}) = \frac{\sqrt{3}}{2\cosh(\tilde{x})^2}\,, \tag{4.32}$$

$$\psi_1(\tilde{x}) = \sqrt{\frac{3}{2}}\,\frac{\sinh(\tilde{x})}{\cosh(\tilde{x})^2}\,. \tag{4.33}$$

The unbounded eigenfunctions are

$$\bar{\psi}^e_{\tilde{k}}(\tilde{x}) = \left(1 + \tilde{k}^2 - 3\tanh(\tilde{x})^2\right)\cos(\tilde{k}\tilde{x}) - 3\tilde{k}\tanh(\tilde{x})\sin(\tilde{k}\tilde{x})\,, \tag{4.34}$$

$$\bar{\psi}^o_{\tilde{k}}(\tilde{x}) = \left(1 + \tilde{k}^2 - 3\tanh(\tilde{x})^2\right)\sin(\tilde{k}\tilde{x}) + 3\tilde{k}\tanh(\tilde{x})\cos(\tilde{k}\tilde{x})\,. \tag{4.35}$$

By adding again two absorbing boundaries at $\tilde{x} = \pm\tilde{L}/2$, we have discrete values of $\tilde{k}$, i. e., $\tilde{k}_{\tilde{L},m}$ for even functions and $\tilde{\kappa}_{\tilde{L},m}$ for odd functions. In the limit $\tilde{L} \to \infty$, we obtain again the classical infinite–square–well relations for eigenstates:

$$\tilde{k}_{\tilde{L}\to\infty,m} = \frac{(2m-1)\pi}{\tilde{L}}\,, \tag{4.36}$$

$$\tilde{\kappa}_{\tilde{L}\to\infty,m} = \frac{2m\pi}{\tilde{L}}\,. \tag{4.37}$$

The normalization constants in this case are

$$\mathcal{N}_e(\tilde{k}, \tilde{L} \to \infty) = \mathcal{N}_o(\tilde{k}, \tilde{L} \to \infty) = \frac{L}{2}(\tilde{k}^2 + 4)(\tilde{k}^2 + 1)\,. \tag{4.38}$$

By applying the same steps as in the case of $\tilde{l} = 1$, we obtain the solution

$$\begin{aligned} p(x,t) &= \frac{3}{4\cosh^4(\alpha x)} + \frac{3}{2}\frac{\sinh(\alpha x)\sinh(\alpha x_0)}{\cosh^4(\alpha x)}e^{-\frac{3}{2}D\alpha^2 t} \\ &+ \frac{\cosh^2(\alpha x_0)}{\pi\cosh^2(\alpha x)}e^{-2D\alpha^2 t}\int_0^\infty \mathrm{d}\tilde{k}\, e^{-\frac{D\alpha^2\tilde{k}^2 t}{2}}\frac{1}{\tilde{k}^2 + 5\tilde{k}^2 + 4}\psi^e_{\tilde{k}}(\alpha x)\,\psi^e_{\tilde{k}}(\alpha x_0) \\ &+ \frac{\cosh^2(\alpha x_0)}{\pi\cosh^2(\alpha x)}e^{-2D\alpha^2 t}\int_0^\infty \mathrm{d}\tilde{k}\, e^{-\frac{D\alpha^2\tilde{k}^2 t}{2}}\frac{1}{\tilde{k}^2 + 5\tilde{k}^2 + 4}\psi^o_{\tilde{k}}(\alpha x)\,\psi^o_{\tilde{k}}(\alpha x_0)\,. \end{aligned} \tag{4.39}$$

The solution for parameters $b = 4$, $D = 2$ and $\alpha = 1$, corresponding to $\tilde{l} = 2$, with the initial condition given by $x_0 = 5$ is shown in Fig. 5 for different time moments t. The evolution of the probability distribution is very similar to that one shown in Fig. 4 for $\tilde{l} = 1$, with the only essential difference that the dynamics is faster and the distribution is somewhat narrower because of a deeper potential well.

5. Conclusions

Using the analogy of the Fokker–Planck equation with the Schrödinger equation, it has been shown how the time–dependent solution can be constructed in the case of mixed eigenvalue spectrum with free and bounded states. The method is based on the idea of introducing two absorbing

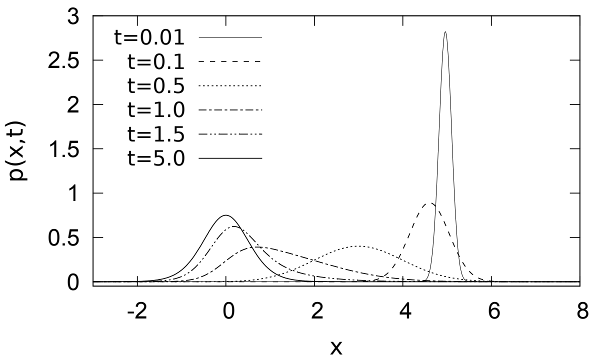

Figure 5. The probability distribution at different time moments t, calculated for the parameters $b = 2$, $D = 4$ and $\alpha = 1$ ($\tilde{l} = 2$) starting at $x_0 = 5$.

boundaries at $x = \pm L/2$, considering the limit $L \to \infty$ afterwards. Although this idea is similar to the one proposed earlier in [10], it is obvious that the problem is quite non-trivial, so that the oversimplified (erroneous) approach of [10] cannot be used – see discussion at the end of Sec. 4.3. Analytical solutions have been found and analyzed in two examples when the Schrödinger potential is constant (constant force) and a shifted Pöschl–Teller potential. For the latter potential, the analytical solutions have been compared with the results of the Crank–Nicolson numerical integration method, and the agreement within an error of 10^{-7} has been found. The time evolution of the calculated probability distribution in these examples is consistent with the usual drift–diffusion dynamics.

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