Model of Vehicular Traffic by Hilliges and Weidlich Revisited

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Summary. Driving of cars on a highway is a complex process which can be described by different means using continuous or discrete basic equations. It always leads to equations of motion with asymmetric interaction.

In 1994 Martin Hilliges and Wolfgang Weidlich from University of Stuttgart developed a phenomenological modeling for dynamic traffic flow in networks, published in 1995 [1]. The authors Hilliges and Weidlich introduce the model in its discrete formulation, carry out a continuous approximation and investigate stationary solutions with respect to stability analytically.

Here we consider the Hilliges–Weidlich–Model once again using our optimal velocity function already introduced in previous papers by Mahnke et al. [2]. We solve the equations of motion given by two coupled partial differential equations numerically and discuss the moving staedy state profiles of density as well as speed. As a first result we present the long-time behaviour. The investitations are still going on and comparisons to related research [3, 4] are in preparartion.

1 The discrete model by Hilliges and Weidlich and its continuous formulation

We have a road segment with some length L. We divide it into N cells with size Δx such that $L = N\Delta x$. In each cell we assume that density ρ_i and velocity v_i are constant (see Fig. 1).

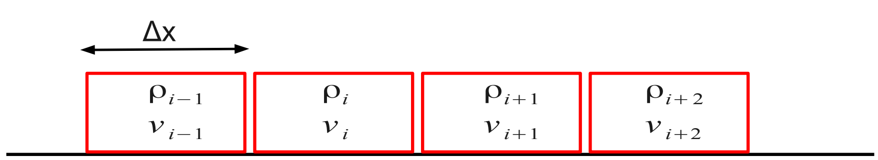

Fig. 1. Road segment with length L divided into cells with length Δx.

The density ρ_i in cell i can change only because of inflow from cell $i-1$ and outflow to cell $i+1$, therefore

$$\frac{\partial \rho_i}{\partial t} \Delta x = j_i - j_{i+1} = \rho_{i-1} v_i - \rho_i v_{i+1} \; . \tag{1}$$

Eq. (1) underlines that traffic is forward oriented. The inflow flux from cell $i-1$ to cell i is defined as $j_i = \rho_{i-1} v_i$.

If we expand density and velocity in Taylor series up to second order at cell i, then

$$\begin{aligned} \rho_{i-1} &= \rho_i - \Delta x \frac{\partial \rho_i}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 \rho_i}{\partial x^2} \\ v_{i+1} &= v_i + \Delta x \frac{\partial v_i}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 v_i}{\partial x^2} \end{aligned} \tag{2}$$

and replace v_i by v and ρ_i by ρ we end up with a partial differential equation for the density

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = \frac{\Delta x}{2} \frac{\partial}{\partial x} \left(\frac{\partial \rho}{\partial x} v - \frac{\partial v}{\partial x} \rho \right) \; . \tag{3}$$

For the velocity we take the well-known relaxation ansatz

$$\frac{Dv}{Dt} = \frac{1}{\tau} \left(V_{opt}(\rho) - v \right) \; , \tag{4}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$ is the material derivative and V_{opt} is a given optimal velocity function. It forms the following set of equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) &= \frac{\Delta x}{2} \frac{\partial}{\partial x} \left(\frac{\partial \rho}{\partial x} v - \frac{\partial v}{\partial x} \rho \right) \; , \\ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} &= \frac{1}{\tau} \left(V_{opt}(\rho) - v \right) \; . \end{aligned} \tag{5}$$

To investigate both equations periodic boundary conditions are chosen. As optimal velocity function we use

$$V_{opt}(\rho) = v_{max} \frac{1}{D^2 \rho^2 + 1} \tag{6}$$

previously introduced by Mahnke et al. [2] to understand car clustering.

By using dimensionless variables $\tilde{x} = \frac{x}{D}$, $\tilde{t} = \frac{t}{\tau}$, $\tilde{\rho} = \rho D$, $\tilde{v} = \frac{v}{v_{max}}$, $\Delta \tilde{x} = \frac{\Delta x}{D}$, $\alpha = \frac{D}{\tau v_{max}}$ the derived set of equations (5) transforms into

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial \tilde{t}} &= -\frac{1}{\alpha} \frac{\partial}{\partial \tilde{x}} (\tilde{\rho} \tilde{v}) + \frac{1}{\alpha} \frac{\Delta \tilde{x}}{2} \frac{\partial}{\partial \tilde{x}} \left(\frac{\partial \tilde{\rho}}{\partial \tilde{x}} \tilde{v} - \frac{\partial \tilde{v}}{\partial \tilde{x}} \tilde{\rho} \right) \; , \\ \frac{\partial \tilde{v}}{\partial \tilde{t}} &= -\frac{1}{\alpha} \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \left(\frac{1}{\tilde{\rho}^2 + 1} - \tilde{v} \right) \; . \end{aligned} \tag{7}$$

2 Temporal development of car cluster solutions

To solve numerically the system of coupled partial differential equations (7) the spatial derivatives are approximated by central finite differences

$$\frac{\partial \tilde{\rho}_i}{\partial \tilde{x}} = \frac{\tilde{\rho}_{i+1} - \tilde{\rho}_{i-1}}{2\tilde{h}} \quad ; \quad \frac{\partial \tilde{v}_i}{\partial \tilde{x}} = \frac{\tilde{v}_{i+1} - \tilde{v}_{i-1}}{2\tilde{h}} \tag{8}$$

$$\frac{\partial^2 \tilde{\rho}_i}{\partial \tilde{x}^2} = \frac{\tilde{\rho}_{i+1} - 2\tilde{\rho}_i + \tilde{\rho}_{i-1}}{\tilde{h}^2} \quad ; \quad \frac{\partial^2 \tilde{v}_i}{\partial \tilde{x}^2} = \frac{\tilde{v}_{i+1} - 2\tilde{v}_i + \tilde{v}_{i-1}}{\tilde{h}^2} \,, \tag{9}$$

where $\tilde{h} = \Delta\tilde{x}$ is a step in space. Then Eqs. (7) together with periodic boundary conditions and given initial conditions are solved using Runge-Kutta 4th order method. Note that by applying central finite differences to the first Eq. of (7) we end up back to Eq. (1).

For some parameter values of α and $\Delta\tilde{x}$ we see that the homogeneous flow solution is unstable and clusters are formed (see Fig. 2). However Eq. (7) describes two types of clusters, those which are moving in the opposite direction to driving direction of cars (see Fig. 3) and in the same direction (see Fig. 4). We get clusters which are moving in the same direction as cars if we start with lower values of space averaged initial density as initial condition. This type of cluster is usually not observed in microscopic traffic models, however, one is able to see this in real traffic data analysis and other traffic models.

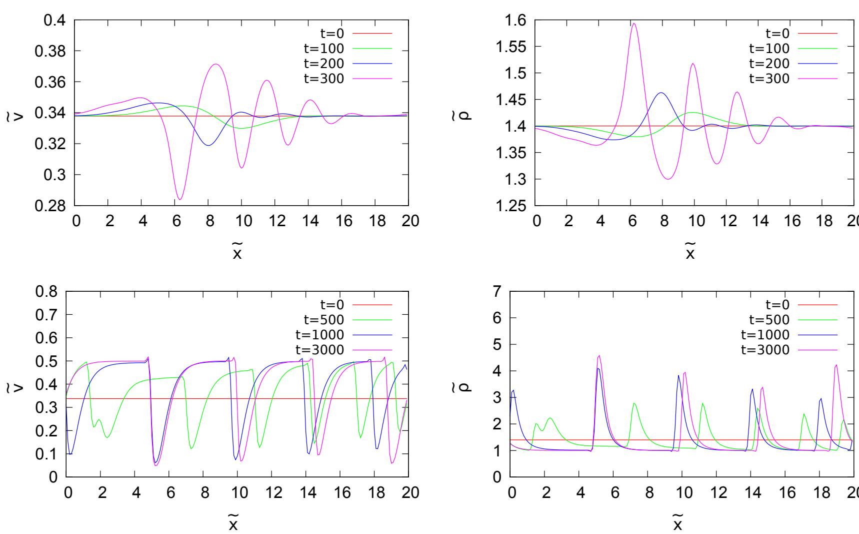

Fig. 2. Solution of Eq. (7) with $\alpha = 4.0$, $\tilde{L} = 20$, $\Delta\tilde{x} = 0.1$ for different time moments starting with the homogeneous solution $\tilde{\rho}(\tilde{t}) = \tilde{\rho}_h = 1.4$, $\tilde{v} = \frac{1}{1+\tilde{\rho}_h^2}$.

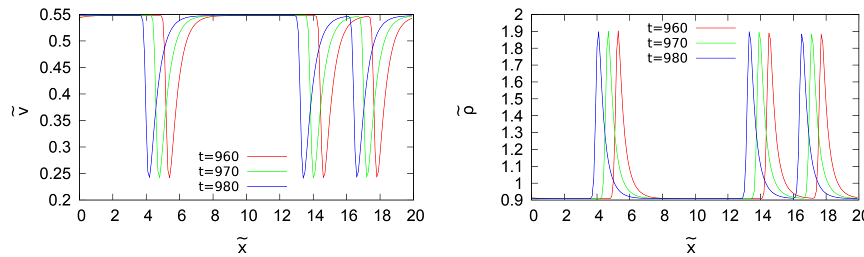

Fig. 3. Clusters moving in the opposite of driving direction of cars (cars drive such that $\tilde{x}$ increases).

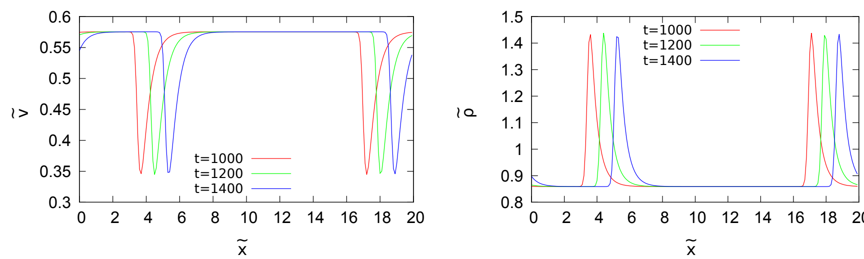

Fig. 4. Clusters moving in the same direction as cars (cars drive such that $\tilde{x}$ increases).

If we tray to simulate Eqs. (7) for longer times, we see, that the number of clusters reduces and after some time there are only few clusters existing. Looking at the density profiles in Fig. 5 it seems that they are going to merge together, but it will take too long computational time. Unfortunately we are still not able to get initial conditions such that a one-cluster-solution can be reached in shorter time.

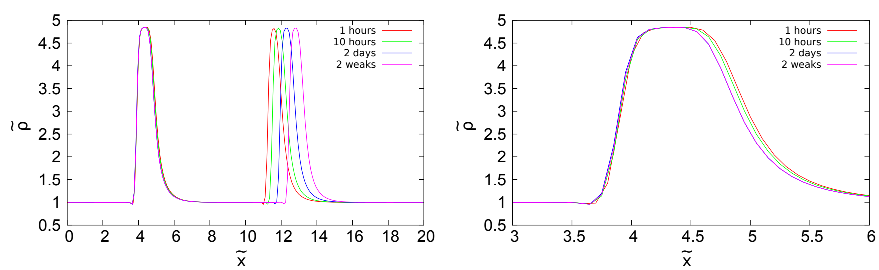

Fig. 5. Long-time solution of Fig. 2 (right-hand part).

3 Remarks on steady states profiles

If we want to get stationary moving profiles which moves with speed $\tilde{v}_g$, then the transformation to a new coordinate system $\tilde{\xi} = \tilde{x} - \frac{1}{\alpha}\tilde{v}_g\tilde{t}$ is useful.

The equations of motion after transformation into moving frame system can be written as

$$
\begin{aligned}
\frac{\partial\tilde{\rho}}{\partial\tilde{t}} - \frac{1}{\alpha}\tilde{v}_g\frac{\partial\rho}{\partial\tilde{\xi}} &= -\frac{1}{\alpha}\frac{\partial}{\partial\tilde{\xi}}(\tilde{\rho}\tilde{v}) + \frac{1}{\alpha}\frac{\Delta\tilde{x}}{2}\frac{\partial}{\partial\tilde{\xi}}\left(\frac{\partial\tilde{\rho}}{\partial\tilde{x}}\tilde{v} - \frac{\partial\tilde{v}}{\partial\tilde{x}}\tilde{\rho}\right) \ , \\
\frac{\partial\tilde{v}}{\partial\tilde{t}} - \frac{1}{\alpha}\tilde{v}_g\frac{\partial v}{\partial\tilde{\xi}} &= -\frac{1}{\alpha}\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{\xi}} + \left(\frac{1}{\tilde{\rho}^2+1} - \tilde{v}\right) \ .
\end{aligned}
\tag{10}
$$

Then for such profiles we have

$$
\frac{\partial\tilde{v}(\tilde{\xi},\tilde{t})}{\partial\tilde{t}} = \frac{\partial\tilde{\rho}(\tilde{\xi},\tilde{t})}{\partial\tilde{t}} = 0 \ . \tag{11}
$$

To calculate such a profile we have to solve the following equations

$$
\begin{aligned}
\frac{\partial\tilde{\rho}}{\partial\tilde{\xi}} &= \frac{2}{\Delta\tilde{x}}\left(\tilde{\rho}\left(1 - \frac{\tilde{v}_g}{\tilde{v}}\right) - \frac{C_g}{\tilde{v}}\right) + \frac{\alpha\tilde{\rho}}{\tilde{v}(\tilde{v}-\tilde{v}_g)}\left(\frac{1}{\tilde{\rho}^2+1} - \tilde{v}\right) \\
\frac{\partial\tilde{v}}{\partial\tilde{\xi}} &= \frac{\alpha}{\tilde{v}-\tilde{v}_g}\left(\frac{1}{\tilde{\rho}^2+1} - \tilde{v}\right) \ ,
\end{aligned}
\tag{12}
$$

where C_g is an integration constant which physical meaning of flux in this moving reference frame $C_g = \tilde{\rho}_h(\tilde{v}_h - \tilde{v}_g)$ and $0 \leq C_g \leq 0.5 - \tilde{\rho}_h\tilde{v}_g$. Here ($\tilde{\rho}_h$, $\tilde{v}_h$) is the homogeneous solution in the stationary reference frame.

Homogeneous solutions

The equations for a homogeneous solution are given as

$$
\begin{aligned}
C_g &= \tilde{\rho}\,(\tilde{v} - \tilde{v}_g) \\
0 &= \alpha\left(\frac{1}{\tilde{\rho}^2+1} - \tilde{v}\right) \ ,
\end{aligned}
\tag{13}
$$

and if we introduce the dimensionless flux $\tilde{Q} = \tilde{\rho}\tilde{v}$ which is given in this case as

$$
\tilde{Q}(\tilde{\rho}) = \frac{\rho}{\rho^2+1} \tag{14}
$$

then Eqs. (13) can be written as

$$
\begin{aligned}
C_g + \tilde{\rho}\tilde{v}_g &= \tilde{Q}(\tilde{\rho}) \ , \\
\tilde{v} &= \frac{1}{\tilde{\rho}^2+1} \ .
\end{aligned}
\tag{15}
$$

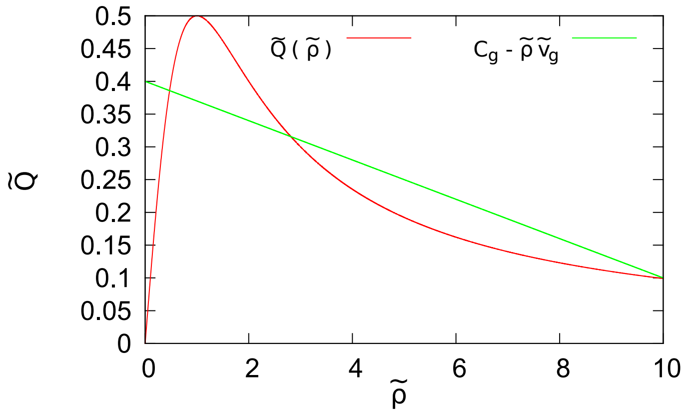

Fig. 6. Graphical solution of Eqs. (15) with $C_g = 0.4$ and $\tilde{v}_g = -0.03$. Solutions are at the intersections of curves.

The Eqs. (15) can be simply solved graphically and an example with $C_g = 0.4$ and $\tilde{v}_g = -0.03$ can be found in Fig. 6. The solutions are the intersection points. Note that for negative $\tilde{v}_g$ the value C_g can be larger then 0.5. In general as we can conclude from Fig. 6 for positive $\tilde{v}_g$ we can get up to two solutions, and for negative $\tilde{v}_g$ up to three solutions. Let us look to a situation with the values $\tilde{\rho}_h = 1.4$ and $\tilde{v}_g = -0.079$. This case corresponds to $C = \tilde{\rho}_h \tilde{v}_h \approx 0.473$ with $C_g \approx 0.584$. Graphical solutions of Eq. 15 can be seen in Fig. 7. As we

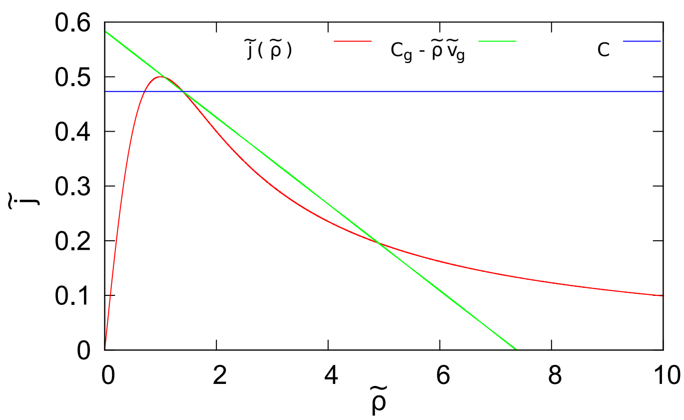

Fig. 7. Graphical solution of Eq. (15) for $C \approx 0.473$ and $\tilde{v}_g = -0.079$. Solutions are at crossings of lines.

can see in Fig. 7 we have 3 solutions $\tilde{\rho} \approx \{\tilde{\rho}_h = 1.4, 1.075, 4.80\}$. We also see that $\tilde{\rho} = \tilde{\rho}_h$ is still a solution of the stationary problem in moving reference frame and it can be shown that this is valid for every velocity $\tilde{v}_g$ and every density $\tilde{\rho}_h$. If we now look to Fig. 5 we actually see that the peak density in clusters is $\tilde{\rho}_c \approx 4.80$ and for free flow $\tilde{\rho}_f \approx 1.08$. So these two non-trivial solutions have some meaning.

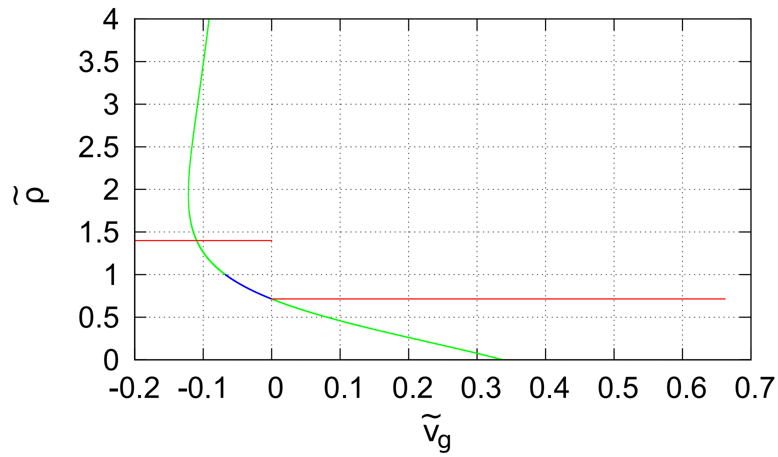

Fig. 8. Valid solution of Eq. (15) as function of $\tilde{v}_g$ for $C \approx 0.473$. Red line shows stationary homogeneous solution, green line shows solutions for which small and large perturbations propagate both with either positive or negative speeds. Blue line shows the solution for which small perturbations propagate with positive speed and large perturbations with negative.

References

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