Air Traffic, Boarding and Scaling Exponents

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Abstract The air traffic is a very important part of the global transportation network. In distinction from vehicular traffic, the boarding of an airplane is a significant part of the whole transportation process. Here we study an airplane boarding model, introduced in 2012 by Frette and Hemmer, with the aim to determine precisely the asymptotic power–law scaling behavior of the mean boarding time $\langle t_b \rangle$ and other related quantities for large number of passengers N. Our analysis is based on an exact enumeration for small system sizes $N \leq 14$ and Monte Carlo simulation data for very large system sizes up to $N = 2^{16} = 65\,536$. It shows that the asymptotic power–law scaling $\langle t_b \rangle \propto N^{\alpha}$ holds with the exponent $\alpha = 1/2$ ($\alpha = 0.5001 \pm 0.0001$). We have estimated also other exponents: $\nu = 1/2$ for the mean number of passengers taking seats simultaneously in one time step, $\beta = 1$ for the second moment of $\langle t_b \rangle$ and $\gamma \approx 1/3$ for its variance. We have found also the correction–to–scaling exponent $\theta \approx 1/3$ and have verified that a scaling relation $\gamma = 1 - 2\theta$, following from some analytical arguments, holds with a high numerical accuracy.

1 Introduction

Recently, following the paper of Frette and Hemmer [1] there has been a spurt of activity regarding airplane boarding, resulting in five papers in Phys. Rev. E [1–5] in roughly 16 months. In the model considered by Frette and Hemmer [1], N passengers have reserved seats, but enter the airplane in arbitrary order ($N!$ possibilities).

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A simplified situation is considered with a single isle of rows and only one seat in each row. It is assumed that a passenger occupies a place equal to the distance between rows. In addition, he or she requires one time step to place carry-on luggage and get seated, the time for walking along the isle being neglected. However, a passenger must wait for a possibility to move forwards to his or her seat if the motion is blocked by other passengers staying or taking seats in front of him or her (see [1] for more details and examples). The number of seats is equal to the number of passengers in this model. In [4], the same process has been considered with more than one seat per row. It has been also discussed there what happens if only some fraction p of the passengers occupy the seats. In a series of works [1,2,4], a non-random ordering of passengers has been also considered. One of the basic quantities of interest is the boarding time t_b of an airplane. All these papers deal with a numerical estimation of the mean boarding time $\langle t_b \rangle$, stating that it is more ore less well consistent with the power law $\langle t_b \rangle = cN^{\alpha}$. Estimates $\alpha = 0.69 \pm 0.01$ and $c = 0.95 \pm 0.02$ have been obtained in [1] from the data with a small number of passengers, $2 < N < 16$.

Later on, it has risen an interesting discussion [2–4] about the value of the exponent α, describing the asymptotic power law at $N \to \infty$. It has been found that the numerical estimates converge to a remarkably different from 0.69 value $\alpha = 1/2$ for large N. In particular, $\alpha = 0.5001 \pm 0.0001$ has been found in [3] from the Monte Carlo simulation data up to $N = 2^{16}$. In fact, $\alpha = 1/2$ is exactly the analytical value reported earlier in [6]. As explained in [4], the $\propto N^{1/2}$ asymptotic behavior follows from the mathematical theorem reported already in [7, 8]. In [6], the proportionality coefficient $c = 4 - 2\ln 2$ has been also found. Corrections to scaling have been considered in [3], as well as in [6]. Numerical estimation in [3] suggests that correction–to–scaling exponent θ in $\langle t_b \rangle = cN^{\alpha}\left(1 + \mathscr{O}\left(N^{-\theta}\right)\right)$ is approximately $1/3$. It has been also numerically found there that the variance of t_b scales with a similar exponent $\gamma \approx 1/3$. In [6] it has been argued that $\alpha - \theta$ is larger than $1/6$, i. e., $\theta < 1/3$. The question about the precise values of θ and γ is interesting and merits further investigation.

2 Exact results for boarding with small number of passengers

Here we consider in some detail the simple model introduced by Frette and Hemmer [1]. For a small number of passengers N, it is possible to consider all $N!$ permutations and count exactly the number of realizations $m(N,t_b)$, corresponding to certain boarding time t_b, where $1 \leq t_b \leq N$, by an appropriate numerical algorithm. The probability to have the boarding time t_b then is $P(N,t_b) = m(N,t_b)/N!$.

The results $P(N,t_b)$ for $N \leq 4$ are collected in Tab. 1 (left). The number of sequences of passengers with increasing seat numbers s is also interesting, since these passengers always get seated simultaneously. This point has been discussed in [1], reporting some exact results. In this case the seats are numbered from left to right, passengers enter the airplane from the left hand side, and we are looking for sequences of passengers also from left to right. A sequence of passengers is repre-

Table 1 The number of realizations $m(N,t_b)$ for boarding of N passengers in t_b time steps (left table), and the number of realizations $m(N,s)$ with s sequences of increasing seat numbers (right table). $P(N,t_b) = m(N,t_b)/N!$ in the left table is the probability that the boarding time is just t_b, and $P(N,s) = m(N,s)/N!$ in the right table is the probability that there are just s sequences.

N	t_b	$m(N,t_b)$	$P(N,t_b)$
1	1	1	1
2	1	1	0.5
	2	1	0.5
3	1	1	0.25
	2	4	0.75
	3	1	0.25
4	1	1	1/24 $\approx$ 0.04167
	2	12	0.5
	3	10	5/12 $\approx$ 0.41667
	4	1	1/24 $\approx$ 0.04167

N	s	$m(N,s)$	$P(N,s)$
1	1	1	1
2	1	1	0.5
	2	1	0.5
3	1	1	0.25
	2	4	0.75
	3	1	0.25
4	1	1	1/24 $\approx$ 0.04167
	2	11	11/24 $\approx$ 0.45833
	3	11	11/24 $\approx$ 0.45833
	4	1	1/24 $\approx$ 0.04167

sented by the corresponding sequence of seat numbers. For example, the sequence 1 2 3 4 represents a queue of $N = 4$ passengers, where the last passenger staying in the queue has the seat number 1, the passenger staying in front of him or her has the seat number 2, and so on. In this case there is only one sequence of increasing seat numbers ($s = 1$) when looking from left to right, which means that all passengers get seated simultaneously in one time step, i. e., the boarding time is $t_b = 1$. A naive guess would be that $t_b = s$. The number of realizations $m(N,s)$, corresponding to certain s, as well as the probability $P(N,s)$ to have just s sequences of passengers with increasing seat numbers, can be easily calculated for a small N.

The results $P(N,s)$ for $N \leq 4$ passengers are collected in Tab. 1 (right). The probability distribution $P(N,s)$ is always symmetric, as it follows from the exact results of [1]. It is seen also in Tab. 1. On the other hand, it is evident from this table that already at $N = 4$ the probability distribution $P(N,t_b)$ is asymmetric, which means that $t_b \neq s$. This effect appears because of merging of the sequences with increasing seat numbers. For $N = 4$ such a merging occurs only for one of $4! = 24$ possible permutations, i. e., for the arrangement 2 1 4 3 with $s = 3$. In this case, the passenger with seat number 1 gets seated simultaneously with the passenger with seat number 3, although these two passengers belong to two different sequences with increasing seat numbers. As a result, two sequences merge after the first step, and the remaining two passengers get seated simultaneously in the second step. It means that $t_b = 2 < s$ holds in this case.

Such cases of merging makes the problem non-trivial and does not allow us to obtain an exact solution for arbitrary N analytically. The number of merging increase very significantly for larger N. The exactly enumerated values of $m(N,t_b)$ and the corresponding values of $P(N,t_b)$ are collected in Tab. 2 for $5 \leq N \leq 13$. In Tab. 3, the results for $N = 14$ are shown, including also those for $m(N = 14, s)$ and $P(N = 14, s)$. The probability distributions $P(N = 14, t_b)$ and $P(N = 14, s)$ are depicted in

Table 2 The number of realizations $m(N,t_b)$ for boarding of N passengers in t_b time steps. $P(N,t_b) = m(N,t_b)/N!$ is the probability that the boarding time is just t_b.

t_b	$m(N,t_b)$	$P(N=5,t_b)$
1	1	0.00833
2	33	0.275
3	66	0.55
4	19	0.15833
5	1	0.00833

t_b	$m(N,t_b)$	$P(N=6,t_b)$
1	1	0.00139
2	88	0.12222
3	372	0.51667
4	227	0.31528
5	31	0.04306
6	1	0.00139

t_b	$m(N,t_b)$	$P(N=7,t_b)$
1	1	0.00020
2	232	0.04603
3	1956	0.38810
4	2218	0.44008
5	586	0.11627
6	46	0.00913
7	1	0.00020

t_b	$m(N,t_b)$	$P(N=8,t_b)$
1	1	0.00002
2	609	0.01510
3	9973	0.24735
4	19587	0.48579
5	8824	0.21885
6	1261	0.03127
7	64	0.00159
8	1	0.00002

t_b	$m(N,t_b)$	$P(N=9,t_b)$
1	1	2.7557×10^{-6}
2	1596	0.00440
3	50236	0.13844
4	163969	0.45185
5	117589	0.32404
6	27006	0.07442
7	2397	0.00661
8	85	0.00023
9	1	2.7557×10^{-6}

t_b	$m(N,t_b)$	$P(N=10,t_b)$
1	1	2.7557×10^{-7}
2	4180	0.00115
3	252299	0.06953
4	1335180	0.36794
5	1460396	0.40245
6	503411	0.13873
7	69057	0.01903
8	4166	0.00115
9	109	0.00003
10	1	2.7557×10^{-7}

t_b	$m(N,t_b)$	$P(N=11,t_b)$
1	1	2.5052×10^{-8}
2	10945	0.00027
3	1268890	0.03179
4	10731205	0.26884
5	17405710	0.43605
6	8630106	0.21620
7	1707964	0.04279
8	155075	0.00388
9	6767	0.00017
10	136	3.4071×10^{-6}
11	1	2.5052×10^{-8}

t_b	$m(N,t_b)$	$P(N=12,t_b)$
1	1	2.0877×10^{-9}
2	28656	0.00006
3	6402738	0.01337
4	85860395	0.17925
5	202624251	0.42301
6	140460107	0.29324
7	38400800	0.08017
8	4898366	0.01023
9	315693	0.00066
10	10426	0.00002
11	166	3.4655×10^{-7}
12	1	2.0877×10^{-9}

t_b	$m(N,t_b)$	$P(N=13,t_b)$
1	1	1.6059×10^{-10}
2	75024	0.00001
3	32435686	0.00521
4	687285783	0.11037
5	2329632160	0.37412
6	2213481380	0.35546
7	811899122	0.13038
8	139225896	0.02236
9	12374938	0.00199
10	595214	0.00010
11	15396	2.4725×10^{-6}
12	199	3.1957×10^{-8}
13	1	1.6059×10^{-10}

Fig. 1, showing also the mean value and the standard deviation for the boarding time distribution.

3 Mapping to the two-dimensional problem of the longest increasing sequence

A passenger sequence can be rendered as a two-dimensional scatter plot. Each passenger is represented by a point with coordinates $x = i/N$ and $y = j/N$, where i is

Table 3 The number of realizations $m(N=14,t_b)$ for boarding of $N=14$ passengers in t_b time steps (left table), and the number of realizations $m(N=14,s)$ with s sequences of increasing seat numbers (right table). $P(N=14,t_b)=m(N=14,t_b)/N!$ in the left table is the probability that the boarding time is just t_b, and $P(N=14,s)=m(N=14,s)/N!$ in the right table is the probability that there are just s sequences for $N=14$.

t_b	$m(N=14,t_b)$	$P(N=14,t_b)$
1	1	1.1471×10^{-11}
2	196417	2.2530×10^{-6}
3	164973584	0.00189
4	5519763360	0.06332
5	26642715539	0.30561
6	34207960967	0.39239
7	16491836851	0.18917
8	3688831863	0.04231
9	432622448	0.00496
10	28312826	0.00032
11	1055151	0.00001
12	21957	2.5186×10^{-7}
13	235	2.6956×10^{-9}
14	1	1.1471×10^{-11}

s	$m(N=14,s)$	$P(N=14,s)$
1	1	1.1471×10^{-11}
2	16369	1.8776×10^{-7}
3	4537314	0.00005
4	198410786	0.00228
5	2571742175	0.02950
6	12843262863	0.14732
7	27971176092	0.32085
8	27971176092	0.32085
9	12843262863	0.14732
10	2571742175	0.02950
11	198410786	0.00228
12	4537314	0.00005
13	16369	1.8776×10^{-7}
14	1	1.1471×10^{-11}

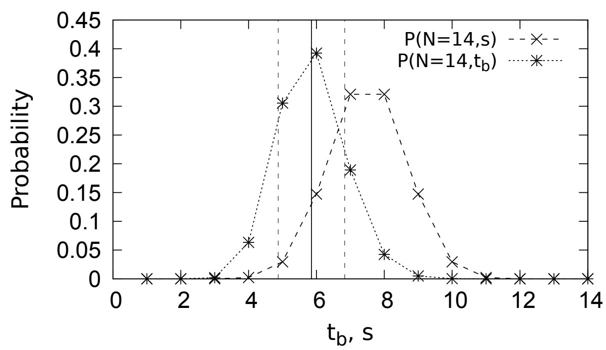

Fig. 1 The probability distributions $P(N,t_b)$ and $P(N,s)$ for $N=14$ passengers. The mean boarding time $\langle t_b\rangle=5.85212$ is shown by a vertical solid line, the range $\pm\sigma$ of one standard deviation $\sigma=0.98116$ from the mean value is indicated by vertical dashed lines.

his/her sequential index in the queue and j is his/her seat number. Note that in this case the passenger, who enters the airplane first has the index $i=1$, the passenger behind him or her has the index $i=2$, and so on. In the asymptotic limit $N\to\infty$, an averaged over ensemble of sequences density of points $(i/N, j/N)$, normalized by N, gives the probability density function $p(x,y)$.

According to the mathematical theorem in [7,8], the length of the longest increasing subsequence asymptotically scales as $N^{1/2}$, provided that $p(x,y)$ is finite. A subsequence $\left\{(x_{i_1},y_{j_1}),(x_{i_2},y_{j_2}),\ldots,(x_{i_l},y_{j_l})\right\}$ of pairs of real numbers with $0\le x_i\le 1$

and $0 \leq y_i \leq 1$ for $i = 1, 2, \ldots, N$ is called an increasing subsequence, if $x_{i_m} < x_{i_{m+1}}$ and $y_{i_m} < y_{i_{m+1}}$ holds for $m = 1, 2, \ldots, l-1$, where i_m is a sequence of non-repeated indices between 1 and N. In the considered here model, the distribution of points in the xy plane is fully random, so that $p(x,y) \equiv 1$ is indeed finite.

The papers [4–6] deal with claim that the length l of the longest increasing sequence is equal to the boarding time t_b. We have check it for $N = 4$, considering the two-dimensional scatter plots for all $4! = 24$ permutations in Fig. 2. In each of the cases, the longest increasing sequence is shown by connecting the points of this sequence by lines. The number of points in this graph is equal to l. In one of the cases no lines are present, implying that $l = 1$. In the cases, where there are several sequences with the same maximal length, different lines are used to distinguish them. In 23 of 24 cases we can see that t_b is indeed equal to the length l of the longest increasing sequence. However, there is one exception, corresponding to the sequence 3 1 4 2 in the notations of Sec. 2. In this case, the seat numbers are $j_1 = 2$, $j_2 = 4$, $j_3 = 1$ and $j_4 = 3$ for passengers numbered sequentially from right to left, as considered in this section. The corresponding scatter plot is the second one in the third row in Fig. 2. Evidently, the boarding time in this case is $t_b = 3$, but $l = 2$.

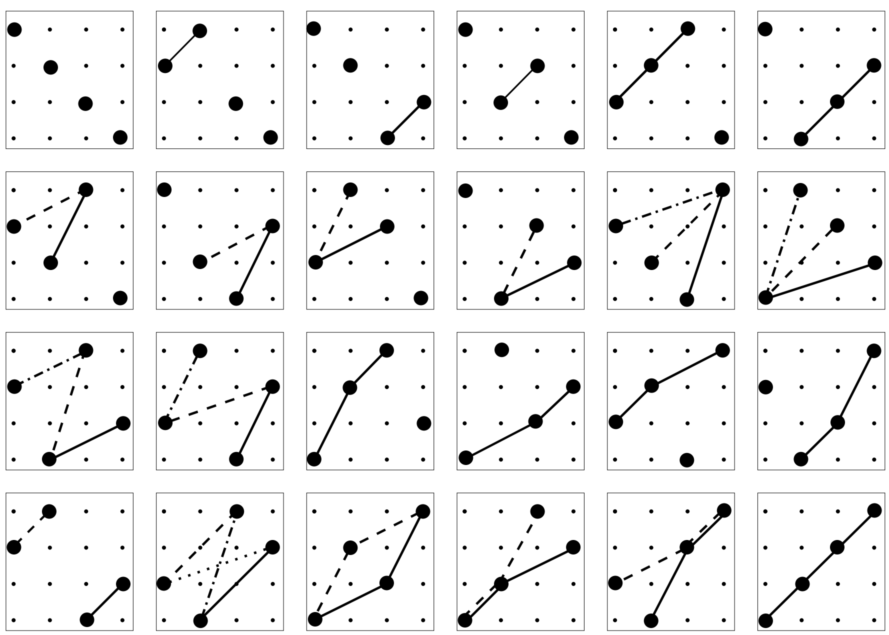

Fig. 2 Scatter plots with horizontal and vertical axes representing the sequential number and the seat number for each of $N = 4$ passengers, plotted by solid circles. The connecting lines are used to show the longest increasing sequences.

This exception shows that the mapping of the original problem to the problem of finding the longest increasing sequence is not exact. Nevertheless, it is possible that the asymptotic scaling of $\langle l \rangle$ and $\langle t_b \rangle$ is described by the same exponent α, e. g., if t_b/l is finite (and nonzero at $N \to \infty$) in a fraction of cases which tends to unity at $N \to \infty$. According to the above mentioned theorem of [7,8], $\langle l \rangle$ scales with the exponent $\alpha = 1/2$ at $N \to \infty$. It is also very plausible that $\langle t_b \rangle$ scales with the same exponent owing to the mentioned here reason, since $\alpha = 1/2$ is accurately confirmed by Monte Carlo simulations [3].

The ensemble of realizations, illustrated in Fig. 2, is unchanged if each of the plots is mirror-reflected with respect to the diagonal $y = x$. The same is true for the mirror–reflection with respect to the other diagonal $y = 1 - x$. Thus, the mirror–symmetric with respect to each other plots appear with equal probability. This is an evident symmetry property for any passenger number N in the considered here mapping, where the number of seats is equal to the number of passengers N and all $N!$ permutations are equally probable. Therefore, if in the asymptotic limit $N \to \infty$ the plot of the longest increasing sequence follows certain curve $y = f(x)$, then there exist also mirror–symmetric curves with respect to both diagonals, representing equivalent plots of increasing sequences of the same (i. e., maximal) length. Hence, the curve $y = f(x)$ is unique only if it follows the diagonal $y = x$ (it cannot follow the other diagonal $y = 1 - x$, since it must be increasing).

Because it turns out that the often used [4–6] and tested here mapping to the problem of finding the longest increasing sequence is inexact, and we also cannot see how the analytical solutions of [5,6] reflect the outlined here symmetry of such a mapping in the simplest case, we mainly rely on our simulation results.

4 Asymptotic scaling results for airplane boarding with large number of passengers

According to [3], the mean boarding time $\langle t_b \rangle$ and its second moment $\langle t_b^2 \rangle$ for large N values about 2^{16} are very accurately described by asymptotic formulas

$$\langle t_b \rangle = A t^{\alpha} \left(1 + a_1 N^{-\theta} + a_2 N^{-2\theta} + o\left(N^{-2\theta}\right)\right) \tag{1}$$

$$\langle t_b^2 \rangle = B t^{\beta} \left(1 + b_1 N^{-\theta} + b_2 N^{-2\theta} + o\left(N^{-2\theta}\right)\right) \tag{2}$$

with the exponents $\beta = 2\alpha = 1$ and $\theta \approx 1/3$. Since the boarding time distribution is asymptotically sharp at $N \to \infty$, the relation $B = A^2$ holds for the coefficients. The exponent $\alpha = 1/2$ agrees with the results of [2,4–6]. The coefficient A has been estimated in [3] (see Fig. 1 there) to be $A = 2.6092 \pm 0.0002$, which is similar to $A = 4 - 2\ln 2 \approx 2.6137$ of [6]. We consider also the variance of the boarding time, which scales as

$$\mathrm{var}(t_b) = \langle t_b^2 \rangle - \langle t_b \rangle^2 \propto N^{\gamma} \tag{3}$$

at large N. According to (1) and (2), where $B = A^2$ and $\beta = 2\alpha = 1$, we have $\gamma = 1 - \theta$ if $b_1 - 2a_1 \neq 0$, and $\gamma = 1 - 2\theta$ if $b_1 - 2a_1 = 0$ and $b_2 - 2a_2 - a_1^2 \neq 0$ hold. Our numerical estimation supports the second possibility, as we find that the relation

$$\gamma = 1 - 2\theta \tag{4}$$

is satisfied within the small error bars of the estimates $\theta = 0.330 \pm 0.001$ (see Fig. 2 in [3]) and $\gamma = 0.343 \pm 0.001$ given in [3]. The agreement of these values with $1/3$, however, is not perfect, and we allow a possibility that $\theta < 1/3$ and $\gamma > 1/3$ hold.

5 Discussions and application

The growing need for mobility through the world shows no sign of slowing down. Applications of stochastic processes to transport problems in a large variety of complex systems, including vehicular and pedestrian traffic, are well known [9, 10]. Here we focus on the air traffic and boarding of an airplane as a significant part of the global transportation process. Our Monte Carlo simulation and analysis is one of numerous applications of stochastic methods to study the behavior of complex systems. From the theoretical point of view, it is tightly related to the power–law scaling and universality problems in many–particle systems. From a practical point of view, it could help to understand the boarding process in order to optimize it.

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