Boarding of Finite-Size Passengers to an Airplane

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Abstract An airplane-boarding model, introduced earlier by Hemmer and Frette, is considered. In this model, N passengers have reserved seats, but enter the airplane in arbitrary order. Here we focus on the blocking relations between passengers. The total boarding time is equal to the longest blocking sequence, represented by a line, connecting points of the two–dimensional q vs r scatter plot. Here $q = i/N$ and $r = j/N$, i and j being sequential numbers of passengers in the queue and their seat numbers, respectively. Such blocking sequences have been studied theoretically by Bachmat. We have developed an algorithm for numerical simulation of the longest blocking sequences, and have compared the results with analytical predictions for $N \to \infty$.

1 Introduction

The growing need for mobility through the world shows no sign of slowing down. Like the vehicular traffic, also the air traffic is a very important part of the global transportation network [1]. A distinguishing feature of air traffic is that a significant part of the total transportation time is related to the boarding of an airplane. Here we study an airplane boarding model, introduced in 2012 by Frette and Hemmer [2]. Following this paper, there has been a spurt of activity regarding airplane boarding,

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resulting in five papers in Phys. Rev. E [2, 3, 4, 5, 6] in roughly 16 months. This problem has been also later discussed during the Traffic and Granular Flow conference in 2013 [7]. In the model considered by Frette and Hemmer [2], N passengers have reserved seats, but enter the airplane in arbitrary order. Besides, there is only a single isle of rows and only one seat in each row. Each passenger occupies a place equal to the distance between rows. In this model, a passenger requires one time step to place carry-on luggage and get seated, the time for walking along the isle being neglected. However, a passenger must wait for a possibility to move forwards to his/her seat if the motion is blocked by other passengers. The number of seats is equal to the number of passengers in this model. In [5], the same process has been considered with more than one seat per row. It has been also discussed there what happens if only some fraction p of the passengers occupy the seats. In a series of works [2, 3, 5, 8], a non-random ordering of passengers has been also considered. One of the basic quantities of interest is the boarding time t_b of an airplane. All these papers deal with a numerical estimation of the mean boarding time $\langle t_b \rangle$, stating that it is more ore less well consistent with the power law $\langle t_b \rangle = cN^{\alpha}$. Estimates $\alpha = 0.69 \pm 0.01$ and $c = 0.95 \pm 0.02$ have been obtained in [2] from the data with a small number of passengers, $2 < N < 16$.

Later on, it has risen an interesting discussion [3, 4, 5] about the value of the exponent α, describing the asymptotic power law at $N \to \infty$. It has been found that the numerical estimates converge to a remarkably different from 0.69 value $\alpha = 1/2$ for large N. In particular, $\alpha = 0.5001 \pm 0.0001$ has been found in [4] from the Monte Carlo simulation data up to $N = 2^{16}$. In fact, $\alpha = 1/2$ is exactly the analytical value reported earlier in [9]. As explained in [5], the $\propto N^{1/2}$ asymptotic behavior follows from the mathematical theorem reported already in [10, 11]. In [9], the proportionality coefficient $c = 4 - 2\ln 2 \approx 2.6137$ has been also found. A similar value 2.6092 ± 0.0002 has been numerically obtained in [4] (see Fig. 1 there). Corrections to scaling have been considered in [4], as well as in [9]. Numerical estimation in [4] suggests that correction–to–scaling exponent θ in $\langle t_b \rangle = cN^{\alpha}\left(1 + \mathscr{O}\left(N^{-\theta}\right)\right)$ is approximately $1/3$. It has been also numerically found there that the variance of t_b scales with a similar exponent $\gamma \approx 1/3$. In [7], some analytical arguments have been provided, suggesting that the scaling relation $\gamma = 1 - 2\theta$ holds. It has been found in [7] that the numerically estimated exponents γ and θ very accurately satisfy this scaling relation, whereas the consistency with the value $1/3$ is not perfect, allowing a possibility that $\theta < 1/3$ holds in reality. Probably, even larger than $N = 2^{16}$ system should be simulated to obtain a reliable numerical estimate of this exponent. In [9] it has been argued that $\alpha - \theta$ is larger than $1/6$, i. e., $\theta < 1/3$. The question about the precise values of θ and γ is interesting and merits further investigation.

2 Blocking sequences

In our current study, we use the following precise definitions of the blocking sequences.

(i) Suppose passenger A takes his/her seat at the n-th time step. We say that passenger A has been blocked by passenger B, if B is the closest passenger in front of A among those ones, which took seat at the $(n-1)$-th time step.
(ii) We depict this blocking relation by drawing an arrow from B to A in the scatter plot (number in queue versus seat number). Nodes and arrows, pointing in certain flow direction, represent a blocking sequence. Its length is equal to the number of nodes. By definition, unconnected nodes are blocking sequences of length 1.

By these definitions, the longest blocking sequences have the length t_b, where t_b is the boarding time. They can be easily deciphered starting from the passengers who get seated at the last time step. The number of such sequences is equal the number of these passengers.

3 Point–like passengers

Let us denote by i_A and i_B the sequential numbers of passengers A and B in the queue. By the definition, we always have $i_B < i_A$ if B is blocking A (i. e., B enters airplane first). In a model with point–like passengers, passenger A can be blocked by passenger B only if $j_B < j_A$, where j_A and j_B are the seat numbers of passengers A and B, respectively. It means that blocking sequences are increasing sequences with arrows always pointing upwards.

However, an arbitrary increasing sequence is not necessarily a blocking sequence we defined here. In our blocking sequence, the seating time increases just by one time step $\Delta t = 1$ when moving forwards by one node along the sequence. To the contrary, we can have $\Delta t \geq 1$ for an arbitrary increasing sequence. The maximal possible length thus is t_b, which corresponds to the case, where the increment of seating time is always $\Delta t = 1$. In such a way, we have $\Omega_{bl} \subset \Omega_{incr}$, where Ω_{bl} is the set of longest blocking sequences and Ω_{incr} is the set of longest increasing sequences. In general, Ω_{incr} contains more elements than Ω_{bl} owing to the fact that many passengers can get their seats simultaneously. However, since all sequences of Ω_{bl} and Ω_{incr} have the same (and maximal possible) length t_b, these sequences are equivalent and look similar for a large system. Namely, it is expected that they follow certain line inside the unit square for normalized quantities $q = i/N$ and $r = j/N$ at $N \to \infty$, where N is the number of passengers. For point–like cars, this line is known to be the diagonal.

4 Finite–size passengers

In the model introduced by Hemmer and Frette passengers are not point–like. They occupy space, which is equal to the distance between seats. For finite–size passengers, $j_B < j_A$ does not necessarily hold, since the blocking can occur via passengers staying between A and B. In the asymptotic limit $N \to \infty$, the condition $j_B < j_A$ (for $i_B < i_A$) is replaced by

$$dr > -dq\, k\alpha(q,r) \tag{1}$$

for $dq > 0$ Here $k = bu/w$, where u is the passenger width, w is the distance between successive rows (in our case – seats), b is the number of passengers per row (in our case $b = 1$) and $\alpha(q,r) = \int_r^1 p(q,z)dz$ with $p(q,r)$ being the probability distribution in $q-r$ plane ($p(q,r) \equiv 1$ for random queue).

In the asymptotic case, the longest sequences obeying the causal relation (1) have the length t_b. (We can conclude it, considering seating times as in the case of point–like passengers). Denoting this set of sequences by Ω, we have $\Omega_{bl} \subset \Omega$, since the blocking sequences satisfy (1) at $N \to \infty$ and also have the length t_b.

According to [6, 8, 9], it is expected that the sequences of set Ω follow certain line in $q-r$ plane. Since $\Omega_{bl} \subset \Omega$, it has to be true also for the set of longest blocking sequences Ω_{bl}. This line L is obtained by maximizing the line integral,

$$\int_L ds \to \max\ , \tag{2}$$

with the conditions that the integration path (line L) goes from $(0,0)$ to $(1,1)$ and belongs to the unit square. Besides, the measure is given by

$$(ds)^2 = 4D^2\, p(q,r)\left[dqdr + k\alpha(q,r)(dq)^2\right], \tag{3}$$

called the Lorentz metric. In fact, one finds that $D = 1$.

By solving the variational problem one finds [9] that L is given by the geodesic line

$$r(q) = C_1 e^{kq} + C_2 e^{2kq} + 1 \tag{4}$$

for $k < \ln 2$ with coefficients C_1 and C_2 determined from the conditions that $r(0) = 0$ and $r(1) = 1$. For $k > \ln 2$, this line does not fit inside the unit square and therefore the path L goes along the border $r = 0$ up to some point $q = q_0$ and then follows the geodesic (4) from $(q_0, 0)$ to $(1,1)$ [9]. Besides, q_0 is such that the geodesic line is tangent to the border $r = 0$ at this point [9]. In such a way, for $k \geq \ln 2$ one finds

$$r(q) = 0 \qquad : \qquad 0 \leq q \leq q_0(k)\ , \tag{5}$$

$$r(q) = -4e^{k(q-1)} + 4e^{2k(q-1)} + 1 \qquad : \qquad q_0(k) \leq q \leq 1\ , \tag{6}$$

where $q_0(k) = 1 - \ln 2/k$. The resulting curve for $k = 1$ is shown in Fig. 1 by solid line. The mean boarding time $t_b = d(k)\sqrt{N}$ has been reported in [9], where

$$d(k) = 2\sqrt{\frac{e^k - 1}{k}} \qquad : \qquad k \leq \ln 2\ , \tag{7}$$

$$d(k) = 2\sqrt{k} + 2(1 - \ln 2)/\sqrt{k} \qquad : \qquad k > \ln 2\ , \tag{8}$$

It corresponds to the length of the $r(q)$ curve in the Lorentz geometry, where distances are measured according to the metric (3).

5 Simulation results and analysis

In order to test the above discussed theoretical predictions, Monte Carlo simulations of the boarding process have been performed in the simple model with $k = 1$, outlined in the beginning of this section. The longest blocking sequences have been determined according to the definitions given in Sec. 2. These sequences, extracted from three different simulation runs with $N = 10^7$ passengers (top), as well as from one simulation run with $N = 10^8$ passengers (bottom) are shown in Fig. 1.

As we can see, the amplitude of random deviations from the theoretical asymptotic curve is still rather large for $N = 10^7$. These deviations are remarkably smaller for $N = 10^8$. It confirms the expected convergence to the analytical solution (5) – (6) at $N \to \infty$.

It is interesting to note that the fluctuations around the geodesic within $q_0(k) \leq q \leq 1$ are remarkably larger in magnitude than those within $0 \leq q \leq q_0(k)$, where the theoretical curve follows the lower border of the $q-r$ square. On the other hand, the fluctuations in the latter region have larger influence on the total boarding time. Indeed, the geodesic maximizes the boarding time and, therefore, a small deviation from it in the form of $\delta \cdot f(q)$, where $\delta \to 0$, produces a deviation of order $\mathscr{O}\left(\delta^2\right)$ in the boarding time t_b. To the contrary, such a deviation from the lower border of the $q-r$ square produces a fluctuation of order $\mathscr{O}(\delta)$ in t_b.

In fact, there is a phase transition in the behavior of the boarding process at $k = \ln 2$, if the parameter k is varied. In the case of point–like passengers, $k \to 0$, the geodesic is just the diagonal $q = r$. Moreover, the deviations of longest blocking sequences from the diagonal is described by the Tracy–Widom distribution, yielding the correction–to–scaling exponent $\theta = 1/3$, which remains valid for $k < \ln 2$. This exponent has to be changed to a smaller value at $k > \ln 2$ due to the fluctuations in the longest blocking sequences (seen in Fig. 1) within $0 \leq q \leq q_0(k)$ [9], which emerge as soon as k exceeds the critical value $\ln 2$.

In view of this fact, it is important to refine our previous estimations of the exponent θ in [4]. As already mentioned in Sec. 1, the number of passengers $N = 2^{16} = 65536$, considered in [4], might be still too small for an accurate estimation of the exponent θ. Indeed, even at $N = 10^7$ the deviations from the theoretical asymptotic behavior in Fig. 1 are rather large. A numerical estimation of θ from the data for much larger number of passengers (e. g., $N = 10^9$) is a challenge for further simulations, which eventually should be based on a faster algorithm of finding longest blocking sequences.

While the scaling of the mean boarding time $\langle t_b \rangle$ is described by the exponent $\alpha = 1/2$, its variance $\mathrm{var}(t_b) = \langle t_b^2 \rangle - \langle t_b \rangle^2$ is described by another exponent γ, i. e., $\mathrm{var}(t_b) \propto N^\gamma$ holds at $N \to \infty$. An idea has been proposed in [7], that the exponents γ and θ obey the scaling relation

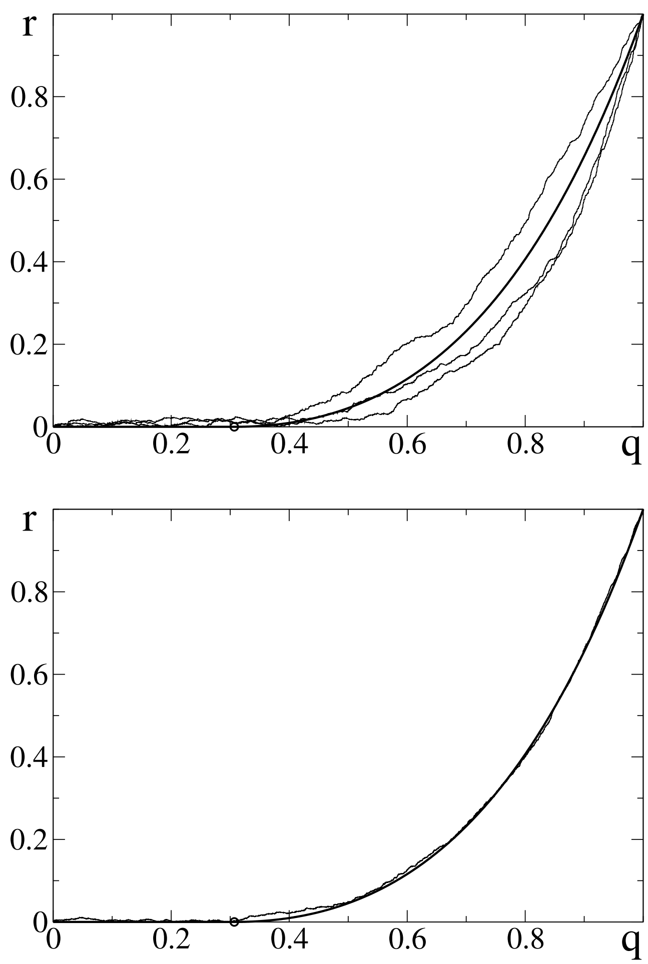

Fig. 1 The analytical curve representing the longest blocking sequences in the asymptotic limit $N \to \infty$ at $k = 1$, given by (5)–(6). The point of departure from border $q_0 = 1 - \ln 2 = 0.30685\ldots$ is marked by a circle. The fluctuating curves represent the longest blocking sequences, extracted from 3 different simulation runs with $N = 10^7$ passengers (top), as well as from one simulation run with $N = 10^8$ passengers (bottom).

$$\gamma = 1 - 2\theta \; . \tag{9}$$

Here we propose a way, which is different from that one in [7], to obtain such a scaling relation. First we note that the variance can be written as

$$\mathrm{var}(t_b) = \left\langle \left(t_b - \langle t_b \rangle \right)^2 \right\rangle \; . \tag{10}$$

Furthermore, the typical values of t_b are smaller than the theoretical asymptotic mean value $\langle t_b \rangle^{as} = d(k)\sqrt{N}$, where $d(k)$ is given by (7)–(8). It is consistent with the fact that $\langle t_b \rangle^{as}$ corresponds to the blocking curve of maximal length (according to the Lorenz metric), so that fluctuations, illustrated in Fig. 1, typically lead to a smaller value of t_b. On the other hand, t_b can be quite close to $\langle t_b \rangle^{as}$. It can be seen from the probability distribution for boarding times $P(t_b)$, illustrated in Fig. 2.

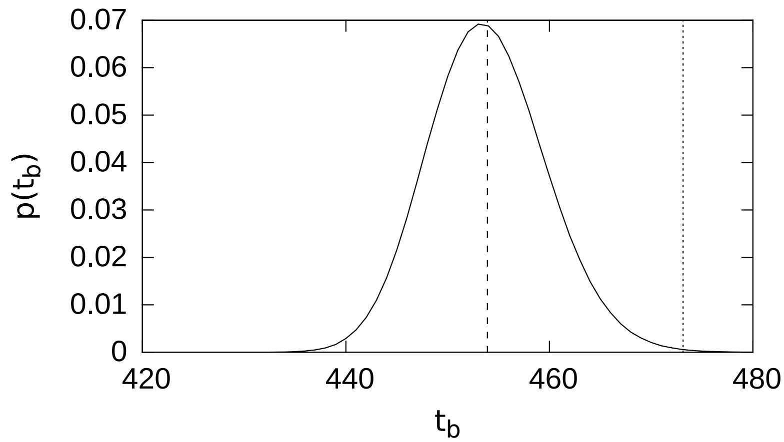

Fig. 2 The probability distribution $P(t_b)$ of the boarding time t_b for the model with $N = 2^{15}$ passengers. The mean boarding time $\langle t_b \rangle = 453.91$ (dashed line) and the asymptotic mean boarding time $\langle t_b \rangle^{as} = 473.13$ (dotted line), corresponding to (7)–(8), are indicated by vertical straight lines.

The probability distribution is shifted below $\langle t_b \rangle^{as}$ in such a way that $\langle t_b \rangle^{as} - \langle t_b \rangle$ is comparable with the width of the distribution. It implies that $\mid t_b - \langle t_b \rangle \mid$ is comparable with $\langle t_b \rangle^{as} - \langle t_b \rangle$ for typical random realizations of the boarding process. Moreover, $\langle t_b \rangle^{as} - \langle t_b \rangle$ scales asymptotically as $\propto N^{\alpha-\theta}$ according to $\langle t_b \rangle = cN^{\alpha}\left(1 + \mathcal{O}\left(N^{-\theta}\right)\right)$ at $N \to \infty$. Hence, typical values of $(t_b - \langle t_b \rangle)^2$ in (10) are comparable with $N^{2(\alpha-\theta)}$ for any large enough N. Consequently, $\mathrm{var}(t_b)$ scales as $\propto N^{2(\alpha-\theta)}$ at $N \to \infty$, i. e., the scaling relation

$$\gamma = 2(\alpha - \theta) \tag{11}$$

holds according to this consideration. Since we have $\alpha = 1/2$, the scaling relation (11) reduces to (9). The actual consideration is sufficiently general, so that this scaling relation is expected to hold both at $k \leq \ln 2$ and $k > \ln 2$.

6 Concluding remarks

In the current study, a simple airplane boarding model, introduced earlier by Frette and Hemmer, has been investigated from different new aspects. In particular, we have focused on the study of blocking relations between the passengers and longest blocking sequences via Monte Carlo simulations and analysis with an aim to test the known theoretical results. We have tackled an important question about the phase transition in the behavior of the boarding process in a generalized model at a certain value of the control parameter k. It is related to some change in the correction–to–scaling exponent θ. Furthermore, we have provided new arguments for the existence of certain universal scaling relation (Eq. (9) or (11)) between the exponents, describing the power–law behavior of the model. Fluctuations in the longest blocking sequences, discussed throughout the paper, allow us to put these phenomena in a general framework of stochastic transport in complex systems [12, 13].

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