# Vehicular Motion on a Hilly Road

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**Abstract** We consider the motion of a point-like car on a one-dimensional hilly road under the influence of gravitation and friction. Based on a Newtonian description we investigate the equations of motion of a particle to discuss the speed and position of a car of mass $m$ on a undulating path $f(x)$.

## 1 Introduction

In the last few decades, many mathematical models for traffic flow have been proposed, in particular *Follow-the-Leader* models. In these models usually the roads are considered to be horizontally flat. But vertically undulated roads, also called sags, are bottlenecks in freeway networks. As pointed out at TRAFFIC AND GRANULAR FLOW '15 that this type of a hilly freeway causes different acceleration behavior of drivers compared with the absence of sags. Traffic flow optimization is important to determine how vehicles should behave at sags in order to minimize total delay [1].
In this contribution we consider the problem of driving at sags from the mathematical point of view based on Newtonian equations of motion. The general solution seems to be well known, but the authors found nothing or references with wrong or incomplete results.

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## 2 The Model

As a starting point, we consider two orthonormal coordinate systems, i. e., $\{x,y\}$ (Cartesian coordinates) and $\{q_T,q_N\}$ (tangent-normal coordinates), which are rotated by an angle $\alpha$ with respect to each other – see Fig. 1. The surface $y = f(x)$, on which the motion takes place, represents a constraint. Therefore the actual coordinates $x$ and $y$, respectively, $q_T$ and $q_N$ are not mutually independent.

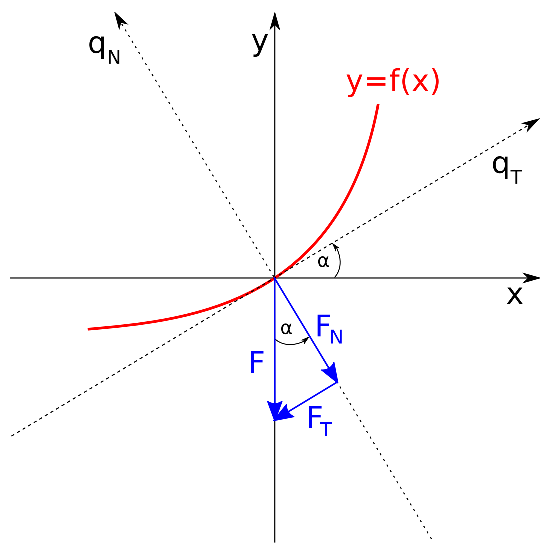


**Fig. 1** The vehicular particle follows the curved path $y = f(x)$ shown as full red line.

## 3 Equations of Motion

The motion of a particle of mass $m$ along a one dimensional path of elevation $y = f(x)$ taking both gravitational and frictional forces into account is given by

$$m\dot{v}_T = F_T + \gamma F_N v_T \ , \tag{1}$$
$$\dot{q}_T = v_T \ , \tag{2}$$

where $q_T, q_N$ are the tangential and normal coordinates to the path defined by $f(x)$, $v_T$ and $\dot{v}_T = dv_T/dt$ are the velocity and acceleration in the direction of motion. The gravitational force $F = -mg$ is split into a part that acts in the direction of motion $F_T = F\sin\alpha = -mg\sin\alpha$ and a part that is normal to the path $F_N = F\cos\alpha = -mg\cos\alpha$ . The Cartesian coordinates $x,y$ and the angle $\alpha$ are defined in the figure above. The coefficient of friction is given as parameter $\gamma \geq 0$.

In Cartesian coordinates the equations of motion (1,2) are

$$m\dot{v}_x = -mg\frac{f'(x)}{1+f'^2(x)} - m\frac{f'(x)f''(x)}{1+f'^2(x)}v_x^2 - mg\gamma\frac{1}{\sqrt{1+f'^2(x)}}v_x\,, \tag{3}$$

$$\dot{x} = v_x\,, \tag{4}$$

where the first derivative of $f$ is $f'(x) = df(x)/dx$ and $f''(x)$ is the second derivative and $f'^2(x)$ is the square of $f'(x)$. The first and second derivatives with respect to time are denoted by $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

The equations (3,4) can be written together as

$$m\ddot{x} = -mg\frac{f'(x)}{1+f'^2(x)} - m\frac{f'(x)f''(x)}{1+f'^2(x)}\dot{x}^2 - mg\gamma\frac{1}{\sqrt{1+f'^2(x)}}\dot{x}\,. \tag{5}$$

A comparison with the results, reported by S. Geisendorf – see Fig. 2, shows the agreement of only the first terms on the right hand side (although the $y$-axis is directed down).

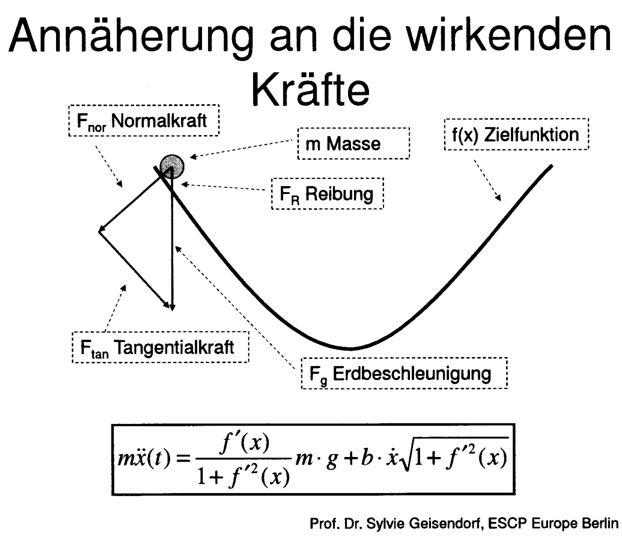


**Fig. 2** From the talk given by Professor Sylvie Geisendorf during the meeting *Physik trifft Volkswirschaftslehre* at the University of Oldenburg, 21.03.2014.

For a motion without friction on a surface without curvature (straight line), only the first term is relevant. The second term appears when the surface is curved, as in the case of the mathematical pendulum (see below), since the curvature contains the second derivative of $f(x)$. The third term describes the friction, which is proportional to coefficient $\gamma$ and velocity $\dot{x}$.

## 4 Derivation of equations of motion

The slope of the function $y = f(x)$ is $f'(x) = dy/dx = \tan\alpha$. Hence, $dq_T = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + f'^2(x)}$ holds due to $dx = dq_T \cos\alpha$ and $dy = dq_T \sin\alpha$.

The gravity force $F = -mg$ is oriented vertically down (in the direction of negative $y$–axis) – see Fig. 1. Its tangential and normal components are $F_T = F \sin\alpha$ and $F_N = F \cos\alpha$, respectively. These components of the force depending on the coordinate $x$ are determined by the relations $\sin\alpha = dy/dq_T = f'(x)/\sqrt{1 + f'^2(x)}$ and $\cos\alpha = dx/dq_T = 1/\sqrt{1 + f'^2(x)}$, yielding

$$F_T(x) = -mg\frac{f'(x)}{\sqrt{1 + f'^2(x)}} \ , \tag{6}$$

$$F_N(x) = -mg\frac{1}{\sqrt{1 + f'^2(x)}} \ . \tag{7}$$

The corresponding to this coordinate velocities read $\{v_x = dx/dt, v_y = dy/dt\}$, respectively, $\{v_T = dq_T/dt, v_N = dq_N/dt\}$. Using the relation $dx = dq_T \cos\alpha$, we obtain

$$v_x = v_T\,\frac{1}{\sqrt{1 + f'^2(x)}} \ . \tag{8}$$

Now we can calculate the acceleration $\dot{v}_x = dv_x/dt = d^2x/dt^2$ in the $x$-direction,

$$\begin{aligned} \dot{v}_x &= \dot{v}_T\,\frac{1}{\sqrt{1 + f'^2(x)}} - \frac{v_T}{\sqrt{1 + f'^2(x)}}\,\frac{f'(x)f''(x)}{1 + f'^2(x)}\,v_x \\ &= \dot{v}_T\,\frac{1}{\sqrt{1 + f'^2(x)}} - v_x^2\,\frac{f'(x)f''(x)}{1 + f'^2(x)} \ . \end{aligned} \tag{9}$$

Multiplying this acceleration by the mass $m$, i. e.,

$$m\dot{v}_x = m\dot{v}_T\,\frac{1}{\sqrt{1 + f'^2(x)}} - mv_x^2\,\frac{f'(x)f''(x)}{1 + f'^2(x)} \ , \tag{10}$$

and replacing $m\dot{v}_T$ by the equation of motion (1), we obtain

$$m\dot{v}_x = \frac{F_T + \gamma v_T\,F_N}{\sqrt{1 + f'^2(x)}} - mv_x^2\,\frac{f'(x)f''(x)}{1 + f'^2(x)} \ . \tag{11}$$

Taking into account the already calculated components of the force as functions of coordinate $x$, i. e., $F_T(x)$ and $F_N(x)$ in Eqs. (6,7), as well as the relation $v_T = v_x\sqrt{1 + f'^2(x)}$, we obtain from Eq. (8) the final result

$$m\dot{v}_x = -\frac{mg\,f'(x)}{1 + f'^2(x)} - \gamma\,\frac{mg\,v_x}{\sqrt{1 + f'^2(x)}} - mv_x^2\,\frac{f'(x)f''(x)}{1 + f'^2(x)} \ . \tag{12}$$

This equation is identical with the equation of motion (3) and therefore also with the Newton's equation (5). By this the derivation is completed.

## 5 Special Situations

We present a solution of eq. (5) in two special cases: motion on an arc of a loop road and on a street with a double well.

The motion of a mass on an arc $y = f(x) = -\sqrt{R^2 - x^2}$ with radius $R$ is described by equations

$$m\dot{v}_x = -mg\frac{\sqrt{R^2-x^2}}{R^2}x - m\frac{x}{R^2-x^2}v_x^2 - mg\gamma\frac{\sqrt{R^2-x^2}}{R}v_x \,, \tag{13}$$

$$\dot{x} = v_x \,, \tag{14}$$

which in polar coordinates $x = R\sin\varphi$ reads

$$\ddot{\varphi} = -\frac{g}{R}\sin\varphi - g\gamma\cos\varphi\,\dot{\varphi} \,. \tag{15}$$

In fact, this is the equation of mathematical pendulum with a new interesting friction term. The solution is illustrated in Fig. 3.

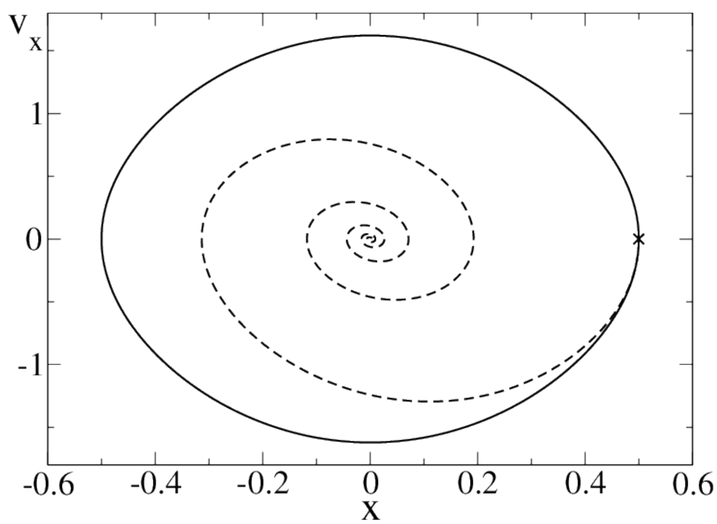


**Fig. 3** Trajectories in the phase space of velocity and coordinate as solutions of eqs. (13,14) with parameters $R = 1$ m, $g = 9.81$ m/s$^2$, $m = 1$ kg for $\gamma = 0$ s/m (thick line) and $\gamma = 0.1$ s/m (dashed line). The initial condition $x(0) = 0.5$ m, $v_x(0) = 0$ m/s is marked by a cross ($\times$).

As another example, we consider the motion on a double-well surface

$$f(x) = \frac{a_2}{2}x^2 + \frac{a_3}{3}x^3 + \frac{a_4}{4}x^4 \,, \tag{16}$$

see Fig. 4, with minima $x_{min}$ located at $-(\sqrt{21}-1)/2 \approx -1.79$ and $+(\sqrt{21}+1)/2 \approx 2.79$.

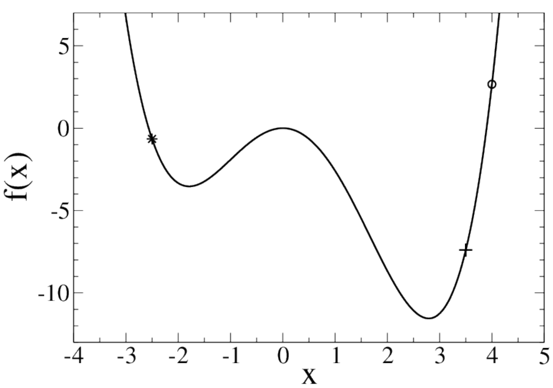


**Fig. 4** The surface (16) with parameters $a_2 = -5$ m$^{-1}$, $a_3 = -1$ m$^{-2}$ and $a_4 = 1$ m$^{-3}$. Various initial coordinates are marked by different symbols (asterisk ($*$), plus ($+$) and open cirlce ($\circ$)).

The coordinate $x(t)$ as function on time is shown in Fig. 5.

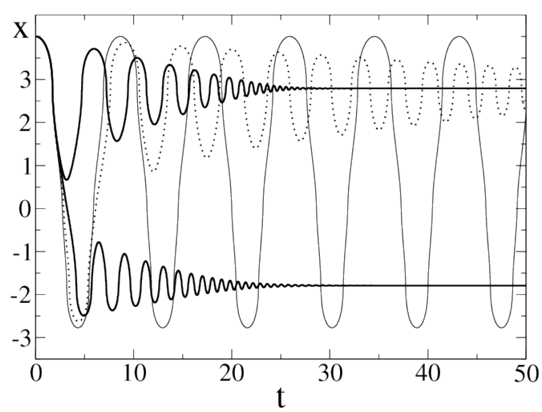


**Fig. 5** Position $x(t)$ (in m) depending on time $t$ (in s) calculated from eqs. (3,4) for the surface (16), shown in Fig. 4, with the parameters $g = 9.81$ m/s$^2$, $m = 1$ kg and initial conditions $x(0) = 4$ m and $v_x(0) = 0$ m/s marked with an open circle ($\circ$) in Fig. 4. The four graphs shown are for the friction coefficients: $\gamma = 0$ s/m (thin line), $\gamma = 0.03$ s/m (dotted line), $\gamma = 0.06$ s/m (lower thick line) and $\gamma = 0.1$ s/m (upper thick line).

The solutions in the phase space of velocity and coordinate is shown in Fig. 6.

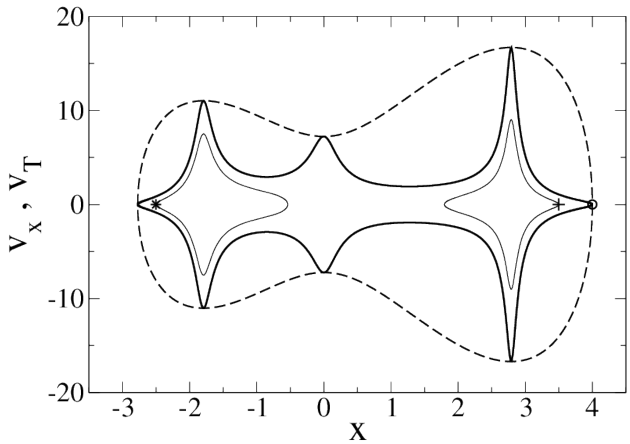


**Fig. 6** Trajectories in the phase space of velocity and coordinate as solutions of eqs. (1,2) with parameters $g = 9.81$ m/s$^2$, $m = 1$ kg and $\gamma = 0$ s/m for $v_x(0) = 0$ m/s and different initial coordinates $x(0)$, shown as in Fig. 4. The dashed line shows the trajectory for $v_T$ at $x(0) = 4$ m, other lines – trajectories for $v_x$ at $x(0) = 4$ m (thick line ($\circ$)), $x(0) = 3.5$ m (right thin line ($+$)) and $x(0) = -2.5$ m (left thin line ($*$)).

## 6 Concluding remarks

The current study improves the understanding of traffic flow at sags from the mathematical point of view. The deterministic description based on Newtonian mechanics is limited for two reasons: (i) We investigate an ideal one-particle system and (ii) we do not include the influence of fluctuations. Randomness is important in a general framework of stochastic transport in traffic systems [2].

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## References

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